Review Article

Comparative Study of Compact Mapping and Hilbert Space

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ABSTRACT

This paper presents the comparative study of compact mapping and Hilbert space. Here, we denote the scalar product of two elements (x, y) of a (real or complex) Hilbert space by (x, y). Here, it is proved in this paper that the comparative study of compact mapping and Hilbert space is the resultant of the spectral theory of compact symmetric operator.

KEYWORDS

Hilbert Space, Compact Mapping, Orthogonal Projection, Scalar Product, Riesz Representation Theorem

1. INTRODUCTION

Hall (1) and Kothe (2,3) are the pioneer worker of the present area . In fact , the present work is the extension of work done by Wong , Yau- Chuen (11) , Prasad, B; et al . (4), Srivastava et al. (5), Srivastava et al. (6) , Srivastava et al .(7), Srivastava, et al. (8), Srivastava et al.(9) and Srivastava et al. (10). In this paper we have studied Comparability about compact mapping and Hilbert Space.

Here , we use the following definitions, Notations and Fundamental ideas :

If M and N are subspaces of a Linear space X such that every $x \in X$ can be written uniquely as $x = y + z$ where $y \in M$ & $z \in N$ then the direct sum of M and N can also be written $X = M \oplus N$ where N is called complimentary subspace of M in X and if $M \cap N = \{0\}$, the decomposition $x = y + z$ is unique.

A given subspace M has many complimentary subspaces and every complimentary subspace of M has the same dimension and the dimension of a complimentary subspace is called co-dimension of M in X , as if $X = R^3$ and M is a plane through the origin then any line through the origin that does not lie in M is a complimentary subspace.

If $X = M \oplus N$ then we define the projection $P: X \to X$ of X on to M along N by $Px = y$, where $x = y+z$ with $y \in M$, $Z \in N$ which is Linear with ran P = M and ker P = N satisfying $P^2 = P$. This property characterizes projections for which the following definitions and theorems follow:-

Definition 1: Any projection associated with a direct sum decomposition of a projection on a Linear space X is a linear map $P:X \to X$ such that $P^2 = P$

Definition 2: An orthogonal projection on a Hilbert space H is also a Linear mapping $P:H \to H$ satisfying $P^2 = P$, $\langle Px,y\rangle = \langle x , Py\rangle$ for all $x, y \in H$.

"An orthogonal projection is necessarily bounded."

Theorem 1 : Let X be a linear space,

(i) If $P:X \to X$ is a projection then $X = \text{ran } P \oplus \ker P$

(ii) If $X = M \oplus N$ where M and N are Linear subspaces of X then there is a projection $P:X \to X$ with ran P = M and ker P = N.

Proof:

For (i) We show that $x \in$ ran P if $x = Px$

If $x = Px$ then clearly $x \in$ ran P

If $x \in$ ran P then $x = Py$ for some $y \in x$

And since $P^2 = P$ which follows that $Px = P^2y = Py = x$

If $x \in$ ran P $\cap$ kerP then x = Px & Px = 0

So ran P $\cap$ kerP = {0}. If $x \in X$ then

We have x = Px + (x- Px) ; where Px $\in$ ran P and (x – Px) $\in$ kerP .

Since P (x- Px) = Px - P^2x = Px –Px = 0

Thus X = ran P $\oplus$ kerP. ………………………(1.1)

Now for (ii)

We consider if X = M$\oplus$ N then $x \in N$ has unique decomposition x = y+z with $y \in M$ & $Z \in N$ and Px = y defines the required Projection.

In particular, in orthogonal subspaces while using Hilbert Space, let us

Suppose that M is a closed subspace of Hilbert Space H then by well-known property we have H = M$\oplus$ $M^{\perp}$. We call the projection of H on to M along $M^{\perp}$ the orthogonal projection of H on to M.

If x= y+z and $x_1 = y_1 + z_1$ where y, $y_1 \in M$ and z, $z_1 \in M^{\perp}$ then by orthogonality of M and $M^{\perp} \Rightarrow$ $<Px, x_1> = <y, y_1 + z_1> = <y, y_1> = <y+z, y_1>$

$= <x, Px_1>$ …………… (1.2)

Which states that an orthogonal projection is self Adjoint. We show the properties (1.1) and (1.2) characterize orthogonal projections with Defn-2 .

Lemma :- If P is a non zero orthogonal projection then || P|| = 1 .

Proof : - If $x \in H$ and $Px \neq 0$ then by Cauchy Schwarz inequality ,

$$\| Px \| = \frac{< Px, Px>}{\| Px \|} = \frac{<x, P^2x >}{\| Px \|} = \frac{< x, Px >}{\| Px \|} \leq \| x\|$$

Therefore || P || ≤ 1. If P ≠ 0 then there is an $x \in H$ with Px ≠ 0 and || P(Px) || = || Px || so that || P|| ≥ 1.

Thus, the Orthogonal Projection P and closed subspace M of H such that ran P = M will must obey one – correspondence, then the kernel of Orthogonal Projection is the Orthogonal Complement of M.

Example .1 – The space L^2 (R) is the Orthogonal direct sum of space M of even functions and the space N of odd functions .The Orthogonal Projection P and Q of H onto M and N, respectively are given by

$$Pf(x) = \frac{f(x) + f(-x)}{2}, \quad Q f(x) = \frac{f(x) - f(-x)}{2}$$

Where I- P = Q .

Proposition:

(a) A Linear functional on a Complex Hilbert space H is a Linear map from H to C. A Linear functional φ is bounded or continuous, if there exists a constant M such that | φ (x) | ≤ M || x || for all x ∈ H .

The norm of bounded linear functional φ is

$$\| \varphi \| = \sup |\varphi (x) |$$

$$\| x \| = 1$$

If $y \in H$ then $\varphi_y (x) = < y, x >$ is a bounded Linear functional on H, with

$$\| \varphi_y \| = \| y\| .$$

(b) If φ is a bounded Linear functional on a Hilbert space H, then there is a unique vector $y \in H$ such that

$$\varphi (x) = < y, x> \quad \text{for all } x \in H$$

Theorem.2: (Riesz representation) If φ is a bounded linear functional on a Hilbert space H , then there is a unique vector $y \in H$ such that

$$\varphi (x) = < y, x> \quad \text{for all } x \in H . \quad \ldots\ldots\ldots\ldots.. \quad (2.1)$$

Proof. If φ = 0 , then y = 0 , so we suppose that φ ≠ 0. In that case , ker φ is a proper closed subspace of H. and , it implies that , there is a nonzero vector

$z \in H$ such that $z \perp \ker\varphi$. We define a linear map P : H $\to$H by

$Px = \varphi (x) / \varphi (z) .z$

Then $P^2 = P$, so Theorem 1 implies that , H = ran P $\oplus$ kerP. Moreover,

ran P ={$\alpha z \mid \alpha \in C$}, kerP = kerφ

So that ran P $\perp$ ker P . It follows that P is an orthogonal projection, and

H ={$\alpha z \mid \alpha \in C$} $\oplus$ kerφ is an orthogonal direct sum. We can therefore write

$x \in H$ as $x = \alpha z + n$, $\alpha \in C$ and $n \in \ker\varphi$.

Taking the inner product of this decomposition with z, we get

$\alpha = < z, x >/ \text{II } z \text{ II}^2$, and evaluating φ on $x = \alpha z + n$, we find that

$$\varphi (x) = \alpha\, \varphi (z).$$

The elimination α from these equations, and a rearrangement of the result

yields φ (x) = < y, x > , where $y = \overline{\varphi (z)}/\text{II } z \text{ II}^2 .z$.

Thus, every bounded linear functional is given by the inner product with a fixed vector. We have already, seen that $\varphi_y (x) = < y, x >$ defines a bounded linear functional on H for every $y \in H$. To prove that there is a unique y in H associated with a given linear functional, suppose that $\varphi_{y1} = \varphi_{y2}$. Then $\varphi_{y1}(y) = \varphi_{y2}(y)$. When $y= y_1- y_2$, which implies that $\text{II } y_1 - y_2 \text{ II}^2 = 0$, so $y_1 = y_2$.

The Map J: H$\to$H* given by $J_y = \varphi_y$, therefore identifies a Hilbert space H with its dual space H*. The norm of φ_y is equal to the norm of y, so j is an isometry.

In this case of complex Hilbert spaces, J is antilinear, rather than linear, because $\varphi_{\lambda y} = \bar{\lambda}\varphi_y$. Thus, Hilbert spaces are self – dual, meaning that H and H* are isomorphic as Banach spaces, and anti-isomorphic as Hilbert spaces. Thus Hilbert spaces are special in this respect. This completes the proof of the Theorem 2.

Proposition: (c) An important consequences of the Riesz representation theorem is the existence of the adjoint of a bounded linear operator on a Hilbert space . The defining property of the adjoint $A^* \in B(H)$ of an operator $A \in B(H)$ is that

$< x, Ay > = <A^*x ,y >$ for all $x, y \in H$ ……… (2.2)

The Uniqueness of A* is obvious. The definition implies that

$(A^*)^* = A, \quad (AB)^* = B^*A^*$.

To prove that A* exists, we have to show that for every $x \in H$, there is a vector

$z \in H$, depending linearly on x such that

$< z, y > = < x, Ay >$ for all $y \in H$ ………….. (2.3)

For fixed x , the map φ_x defined by, $\varphi_x (y) = < x, Ay >$

is a bounded linear functional on H , with $\|\varphi_x\| \le \|A\| \|x\|$. By the Riesz representation Theorem, there is a unique $z \in H$ such that $\varphi_x (y) = < z, y >$. This z satisfies (2.3), So we get $A^*x = z$. The linearity of A* follows from the uniqueness in the Riesz representation theorem and the linearity of the inner product .

Thus, from above definitions, Theorems, Leema, example, Propositions (a), (b), & (C), which shows the proof of the main result as "the representation of compact mappings of Hilbert Spaces is a Consequence of the Spectral theory of Compact symmetric operators.

1) Let H_1 , H_2 be Hilbert spaces , $A \in £ (H_1,H_2)$ compact and not of finite rank . Then, there exists orthonormal systems, e_n , n = 1,2,………. In H_1 and $\{f_n\}$, n= 1,2,…….. in H_2 such that

2) $$A x = \sum_{n=1}^{\infty} \lambda_n (x, e_n) f_n , x \in H_1 \text{ where } \lambda_n > 0 \text{ and } \lambda_n \to 0.$$

Proof: - Since A is Compact, A*A is Compact too and positive, where A* denotes the adjoint in the sense of the scalar product . It follows from Spectral theory that there exists an orthonormal sequence of eigen vectors en , n= 1,2,3,…….. and eigen values $\lambda_n^2 > 0$, $\lambda_n^2 \to 0$ such that

$$A^* A_x = \sum_{n=1}^{\infty} \lambda_n^2 (x, e_n) e_n ,$$

A*A is zero on the orthonormal or complement H of the closed subspace spanned by all the e_n . But then A is zero too on H.

Take $y \in H$ and suppose $Ay \neq 0$.

Then $(A_y , A_y) = (y, A^*A_y) \neq 0$. But this would imply $A^*A_y \neq 0$, Therefore we have a representation

$$A_x = \sum_{n=1}^{\infty} \lambda_n (x, e_n) A e_n.$$

We now define

$f_n = (1/ \lambda_n)A e_n$. Then

$$A_x = \sum_{n=1}^{\infty} \lambda_n (x, e_n) f_n$$

and other proposition will be proved if we Show that $\{ f_n \}$ is an orthonormal systems.

But $(f_i ,f_k) = (\lambda_i^{-1} Ae_i , \lambda_k^{-1} A e_k)$

$= \lambda_i^{-1} \lambda_k^{-1}(A^*Ae_i,e_k)$

$= \lambda_i^{-1} \lambda_k^{-1} (\lambda_i^2 e_i, e_k)$

$=\delta_{ik}$

(3) Conversely every mapping $A \in £(H_1,H_2)$ which has a representation (2) with

$\lambda_n > 0$, $\lambda_n \to 0$ is compact.

Let A_k be $\sum_{n=1}^{K} \lambda_n (x, e_n)f_n$, $\|(A - A_n)x\|^2$

$$\le \sum_{n=k+1}^{\infty} \lambda_n^2 \, |(x , e_n)|^2$$

$\le \epsilon^2 \| x \|^2$, it is $|\lambda_n| \le \epsilon$ for $n > k(\epsilon)$.

Thus A is compact as the limit of A_n in $£_b (H_1 , H_2)$. From this proof and (1) follow immediately.

(4). Let H_1, H_2 be Hilbert Spaces. Then every compact $A \in £_b (H_1, H_2)$ is the limit of a sequence of mappings of finite rank.

Then λ_n of (2) are called the singular values of A and the non- increasing sequence of all singular values of A is uniquely determined by A , the representation (2) can be written in a different way using linear forms instead of scalar product for the coefficients of the f_n .

The scalar product (x,y) in Hilbert space H is linear in x for y fixed , thus it defines a linear functional, $< \bar{y} , x > = (x,y)$, where $\bar{y}$ is uniqually determined . One calls $\bar{y}$ the Conjugate element to y. There exists an Orthonormal basis $\{e_\alpha\}$, $\alpha \in A$, of H such that

For $x = \sum_\alpha \xi_\alpha e_\alpha$, y

$= \sum_\alpha \eta_\alpha e_\alpha$

$(x,y) = \sum_\alpha \xi_\alpha \bar{\eta}_\alpha = < \bar{y} , x >$

Since this is true for all x ϵ H, if follows that $\bar{y} = \sum_{\alpha} \bar{\eta}_\alpha e_\alpha$; the coefficients of $\bar{y}$ are the Conjugate of the coefficients of y.

Hence the Result.

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