Weakly Compatible Mappings for Integral Type Contractive Condition in Probabilistic Metric Space

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Abstract: - In this paper, we prove a common fixed point theorem for pair of weakly compatible mappings satisfying a contractive condition of integral type in probabilistic metric space akin to metric space which generalize the results of Branciari [1] and various results of metric and probabilistic metric spaces. We also provide answer to an open question of Rhoades [14, p-242] in the setting of probabilistic metric spaces. At the end, we provide an example in support of our theorem.

Keywords: Weakly compatible mappings, Fixed point, Probabilistic metric space.

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I. INTRODUCTION

In metric space, for any two points in the space, there is defined a positive number called the distance between the points. In fact, it is suitable to look upon distance concept as a statistical or probabilistic rather than deterministic one, because the advantage of a probabilistic approach is that it permits from the initial formulation a greater flexibility rather than that offered by a deterministic approach. The idea thus appears that, instead of a single positive number, we should associate a distribution function with the point pairs. Thus, for any p, q elements in the space, we have a distribution function F(p,q;x) and interperate F(p,q;x) as the probability that distance between p and q is less than x.

The concept of a probabilistic metric space (Menger space) corresponds to the situations when we do not know the distance between the points, i.e., the distance between the points is inexact rather than a single real number, we know only probabilities of possible values of this distance. Such a probabilistic generalization of a metric space appears to be well adopted for the investigation of physical quantities and physiological thresholds. Thus in 1942, Menger[3] introduced the notion of probabilistic metric space or statistical metric space which is in fact, a generalization of metric space and the study of these spaces were expanded rapidly with the pioneering works of Schweizer-Sklar [7]. The theory of probabilistic metric space is of fundamental importance in probabilistic function analysis.

Further, V.M. Sehgal [8] initiated the study of fixed point in probabilistic metric space (PM-Space). Thereafter, several authors have studied the existence of fixed point in PM-space for two or three or four or sequences of maps satisfying some contractive type conditions.

First, we recall that a real valued function defined on the set of real numbers is known as a distribution functions if it is non-decreasing, left continuous and inf f(x) = 0 , sup f(x) = 1. In what follows H(x) denotes the distribution function defined as follows.

$$H(x) = \begin{cases} 0 \text{ if } x \le 0 \\ 1 \text{ if } x > 0 \end{cases}$$

Definition 1.1: A probabilistic metric space (PM-space) is a pair (X, F) where X is a set and F is a function defined on $X \times X$ into the set of distribution functions such that if x, y and z are the points of X, then

(i) F(x, y; 0) = 0

(ii) F(x, y; t) = H(t) iff x = y

(iii) F(x, y; t) = F(y, x; t)

(iv) if F(x, y; s) = 1 and F(y, z; t) = 1, then F(x, z; s + t) = 1 for all x, y, z ε X and s, t > 0.

For each x and y in X and for each real number $t \ge 0$, F(x,y; t) is to be thought of as the probability that the distance between x and y is less than t. Of course, any metric space (X, d) may be regarded as a PM-space. Indeed, if (X, d) is a metric space, then the distribution function F(x, y; t) defined by

F(x, y; t) = H (t−d(x, y)) induces a PM-space, where d is an usual metric.

Definition 1.2: A t-norm is a 2-place function Δ: $[0,1]\times[0,1] \to [0,1]$ satisfying the following

i) Δ(0, 0) = 0,

ii) Δ(0, 1) = 1,

iii) Δ(a, b) = Δ(b, a),

iv) if a ≤ c, b ≤ d, then Δ(a, b) ≤ Δ(c, d)

v) Δ (Δ(a, b),c) = Δ(a, Δ(b, c)) for all a, b, c in [0, 1].

Definition 1.3.: A Menger PM-space is a triplet (X, F, Δ) where (X, F) is a PM-space and Δ is a t-norm with the following condition:

F(x, z; s + t) ≥ Δ(F(x, y; s), F(y, z; t)) for all x, y, z in X and s, t > 0 .

This inequality is known as Menger's triangle inequality.

In 1966, the notion of contraction mappings on PM-space was first introduced by Seghal. Moreover, "every contraction mapping on a complete Menger space has a unique fixed point". For more detail see, [3, 4, 7, 8].

Definition 1.4: Let (X, F) be a PM-space and f : X → X be an arbitrary mapping on X. Then f is called a contraction if there exist k ε (0, 1) such that for x, y in X and t > 0 the following relation holds.

F(fx, fy; kt) ≥ F(x, y; t).

In 1986, Jungck [2], gave the more generalized concept of compatibility than commutativity and weak commutativity in metric space and proved common fixed point theorems. The study of common fixed points of compatible mapping emerged as an area of vigorous research activity ever since Jungck introduced the notion of compatibility as follows:

A pair of maps f and g:(X,d)→(X,d) is said to be compatible if limn→∞d(fgxn,gfxn) = 0, whenever {xn} is a sequence in X such that limn→∞fxn = limn→∞gxn = t for some t in X

In 1991, Mishra [4] introduced the concept of compatible mappings in PM-space akin to concept of compatibility in metric space introduced by Jungck[2]. However, the study of common fixed points of noncompatible mappings in metric space has been initiated by Pant [5].

In 1994 Pant [5] introduced the concept of R-weakly commuting of mappings as follows:

Definition 1.5[12] Let A and B mappings from a probabilistic metric space (X,F,Δ) into itself. Then A and B are said to be compatible if $\lim_{n\to\infty}$F(AB xn, BA xn, t) = 1, whenever {xn} is a sequence in X such that $\lim_{n\to\infty}$ Axn = $\lim_{n\to\infty}$ Bxn = u for some u > X and for all t > 0. Clearly, weakly commuting mappings are compatible, but the converse may not be necessarily true.

In 1998, Jungck and Rhoades[9] introduced the concept of weakly compatible maps as follows:

A pair of maps f,g:(X,d)→(X,d) is weakly compatible pair if they commute at coincidence points i.e., fx = gx iff fgx = gfx.

Example 1.1[14]. Let X = [0,3] be equipped with the usual metric space d(x, y) = |x−y| . Define f, g : [0, 3] : [0, 3] by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1) \\ 3 & \text{if } x \in [1,3] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3-x & \text{if } x \in [0,1) \\ 3 & \text{if } x \in [1,3] \end{cases}$$

Then for any x ε [1,3], fgx = gfx, showing that f, g are weakly compatible maps on [0,3].

Example 1.2[14]. Let X = R and define f,g : R → R by fx = x/3, x ε R and gx = x2, x ε R. Here 0 and 1/3 are two coincidence points for the maps f and g. Note that f and g commute at 0, i.e. fg(0) = gf(0) = 0, but fg(1/3) = f(1/9) = 1/27 and gf(1/3) = g(1/9) = 1/81 and so f and g are not weakly compatible maps on R.

Remark 1.3[14]. Weakly compatible maps need not be compatible.

In 2002, Branciari [1] proved the following fixed point theorem in metric space:

Theorem 1.1. Let (X,d) be a complete metric space, c e [0,1), f: X ® X a mapping such that, for each x, y e X,

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le c \int_0^{d(x,y)} \varphi(t)\,dt,$$

Where φ: R+→ R+ is a Lebesgue-integrable mapping which is a summable, non-negative, and such that, for each >0, φ(t) dt > 0. Then f has a unique fixed point z ε X

Such that, for each x ε X, limn→∞f ^{n}x = z.

In this paper we use the concept of weakly compatibility mapping satisfying a general contractive condition of integral type in probabilistic metric space by defining

$$F(x,y,t) = \frac{t}{t + d(x,y)} \quad \text{for all x,y e X and for each t > 0.}$$

Now we prove the following result in PM-space:

II. MAIN RESULT

Theorem 2.1: Let (X,F, Δ) be a complete probabilistic metric space, f and g are weakly compatible self maps of X satisfying the following conditions:

(2.1) f(X) g(X),

(2.2) any one of f(X) or g(X) is complete,

(2.3) $$\int_0^{1-F(fx,fy,t)} \varphi(p)\,dp \le c \int_0^{1-F(gx,gy,t)} \varphi(p)\,dp,$$

for each x,y e X, t >0, c e[0,1)

Where φ: R+→ R+ is a Lebesgue- integrable mapping which is a summable, non-negative, and such that

$$\int_0^{\in} \varphi(p)\, dp > 0 \quad \text{for each } >0. \qquad (2.4)$$

Then f and g have a unique common fixed point.

Proof: Let x0 e X. Since f(X) g(X), choose x1 e X such that gx1 = fx0. In general, choose xn+1 such that yn = gxn+1= fxn.

For each integer n≥1, from (2.3),

$$\int_0^{1-F(y_n,y_{n+1},t)} \varphi(p)\, dp = \int_0^{1-F(fx_n,fx_{n+1},t)} \varphi(p)\, dp \le c \int_0^{1-F(gx_n,gx_{n+1},t)} \varphi(p)\, dp = c \int_0^{1-F(fx_{n-1},fx_n,t)} \varphi(p)\, dp$$

$$\le c2 \int_0^{1-F(gx_{n-1},gx_n,t)} \varphi(p)\, dp$$

In this fashion one obtains

$$\int_0^{1-F(y_n,y_{n+1},t)} \varphi(p)\, dp \le c^n \int_0^{1-F(y_0,y_1,t)} \varphi(p)\, dp$$

Taking the limit as n→∞, we have

$$\int_0^{1-F(y_n,y_{n+1},t)} \varphi(p)\, dp = 0.$$

which, from (2.4), implies that

limn→∞F(yn,yn+1,t) = 1 (2.5)

We now show that {yn} is a Cauchy sequence. Suppose that it is not. Then there exists an ∈>0 and subsequences {m(q)} and {n(q)} such that m(q)<n(q)<m(q+1) with

1-F(ym(q),yn(q),t1+t2) ≥ , 1-F(ym(q),yn(q)-1,t1+t2) < (2.6)

And from (2.5) limq→∞F(yn(q)-1,yn(q),t) = limq→∞F(ym(q)-1,ym(q),t) = 1

Using (M4), we have

F(ym(q)-1,yn(q)-1,t1+t2) ≥ F(ym(q)-1,ym(q),t1)* F(ym(q),yn(q)-1,t2)

> F(ym(q)-1,ym(q),t1)*1-

or 1-F(ym(q)-1,yn(q)-1,t1+t2) < as q→∞.

Therefore, from (2.3) and (2.6),

$$c \int_0^{1-F(y_{m(q)-1},y_{n(q)-1},t_1+t_2)} \varphi(p)\, dp \ge \int_0^{1-F(y_{m(q)},y_{n(q)},t_1+t_2)} \varphi(p)\, dp \ge \int_0^{\in} \varphi(p)\, dp.$$

which implies $c \int_0^{\in} \varphi(p)\, dp \ge \int_0^{\in} \varphi(p)\, dp$ as q→∞

a contradiction, since c e[0,1). Therefore, {yn} is a Cauchy sequence in X. Since g(X) is complete, so there exists a point z e g(X) such that limn→∞yn = limn→∞ gxn = z. Since z e g(X) therefore, there exists a point u e X such that gu = z. Now we show that z = fu = gu. If possible fu ≠ gu, by inequality (2.3), we have

$$\int_0^{1-F(fx_n,fu,t)} \varphi(p)\, dp \le c \int_0^{1-F(gx_n,gu,t)} \varphi(p)\, dp, \quad \text{for}$$

each x,y e X, c e [0,1).

Letting n→∞, we get

$$\int_0^{1-F(gu,fu,t)} \varphi(p)\, dp \le c \int_0^{1-F(gu,gu,t)} \varphi(p)\, dp,$$

which implies fu = gu.

Since f and g are weakly compatible, therefore , it follows that fz = fgu =gfu =gz and then ffu= fgu =gfu =ggu. Now our aim to show that z= fu = gu is a common fixed point of f and g. If possible fu ≠ ffu.

Using (2.3) again, we obtain

$$\int_0^{1-F(fu,ffu,t)} \varphi(p)\, dp \le c \int_0^{1-F(gu,gfu,t)} \varphi(p)\, dp = c \int_0^{1-F(fu,ffu,t)} \varphi(p)\, dp,$$

which is a contradiction, since c e [0,1), this implies fu = ffu =gfu and therefore, fu is a common fixed point of f and g.

Uniqueness: Suppose that w is also a common fixed points of f and g.

Then from (2.3), we have

$$\int_0^{1-F(z,w,t)} \varphi(p)\, dp = \int_0^{1-F(fz,fw,t)} \varphi(p)\, dp \le c \int_0^{1-F(gz,gw,t)} \varphi(p)\, dp = c \int_0^{1-F(z,w,t)} \varphi(p)\, dp,$$

which implies that z = w, and hence the common fixed point is unique.

Example 2.1. Let X=[3,22] and d be usual metric on X. Let ,g: X→X be defined by if x = 3 or x > 7 ; if 3 < x ≤ 7,g3 = 3 ; gx = 10 if 3 < x ≤ 7 ; gx = (x+2)/3 if x > 7 and ψ, φ: [0,∞)→[0,∞), where ψ(t) = (t+1)t+1-1 and φ(t) =

Also let us define

$F(x,y,t) = \frac{t}{t + d(x,y)}$ for all x,y e X and for each t > 0.

Moreover, fX = {3} {8}, gX = [3,8] {10}. Hence fX gX. To see that f and g are non-compatible maps, consider the sequence {xn = 7+ 1/n , n≥1} in X. Then limn→∞fxn = 3, limn→∞gxn = 3, limn→∞fgxn = 8 and limn→∞gfxn = 3. Hence f and g are noncompatible maps. But they are weakly compatible since they commute at coindence point at x = 3.

Thus and satisfy all the conditions of the above theorem and have a unique common fixed point at x = 3.

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