# Existence of Solution of Fractional Voltraa-Integro-Differential Equations

V. Rajadurai[1], S. Kumara Ghuru[2]

*[1]Assistant Professor- Government Arts and Science College –Coimbatore*

*[2]Assistant Professor- Chikkana Government Arts and Science College –Tirupur*

***Abstract: The aim of this paper is to the existence of solutions of Fractional Volterra Integro-differential equations by using Laplace decomposition method. The tools employed in the solving the equations based on Laplace transform technique for non-linear equations. Further Laplace decomposition method apply to find the approximate solution of non-linear integro-differentiated equation. The fractional derivative is described in the caputo sense. Finally, this method is illustrated by some examples presented.***

***Keywords: Volterra Integro-differential equations Laplace Transform, Adomion Polynomial, Fractional derivatives estimates on solutions.***

## I. INTRODUCTION

The Laplace decomposition method is a powerful numerical algorithm to solve non-linear ordinary and partial Integro-differential equations. Recently, several numerical methods to solve freefind integro-differential equations have been studied. Dretham and Ford [3] presented the analysis of fractional differential equation. Also, Momani and Qaralleh [6] applied Adomian Polynomials to solve fractional integro-differential equations. X.Ma and Huang [9] discussed Numerical solution of fractional Integro-differential equations by hybrid collocation method. Zhang and Tang [7] presented homotopy analysis method for higher-order fractional integro-differential equations.

Most of the non-linear fractional integro-differential equations do not have exact analytic solutions, so only numerical and approximate technique used. The analytic results on existence and uniqueness of problems solutions to fractional integro-differential equations have been studied by many researchers.

In this paper, we study the Laplace decomposition method for a special kind of non-linear fractional integro-differential equations.

$$D^{\alpha} y(t)=r(t)y(t)+g(t)+\lambda\int_0^t k(t,s)\, f[y(s)]ds \qquad (1)$$

and

$$D^{\alpha} y(t)=r(t)y(t)+g(t)+\lambda\int_0^1 k(t,s)\, f[y(s)]ds \qquad (2)$$

for $t \in [0,1]$ with the initial conditions.

$y(i) = \delta i$, $i = 0,1,2\ldots. n-1$, $n-1 < \propto \leq \triangleright n$, $n \in N$

where $g \in L2([0,1])$, $r \in L2([0,1])$, $K \in L2([0,1]2)$

are known functions. y(t) is unknown functions. $D\alpha$ is the Caputo differential Operator of order '$\alpha$'

In this paper, we applied Laplace Transform and Adomian Polyomials to solve non-linear integro differential equation of fractional order.

## II. PREMILINARIES

*Basic Definitions*

**Definition 1** The Laplace Transform of a real valued function f(t), t > 0, is defined as

$$L\ [f(t)] = \int_0^{\infty} e^{-st} f(t)dt = F(s),$$ where S is real or complex valued parameter.

**Definition 2** Laplace Transform of an integral

If L[f(t)] = F(S), then

$$L\ [\textstyle\int_0^t f(t)dt] = \frac{1}{S} L[f(t)]$$

**Definition 3** Convolution of two functions f(t) and g(t) is defined as

$$(f * g)(t) = \int_0^t f(u)\, g(t-u)du$$

**Definition 4** Laplace Transform of convolution Theorem

L[f * g) = L[f(t) L [g(t) = F(s) G (s)

**Definition 5** The fractional derivative $D^{\gamma}$ of f(t) in the Caputo's sense is defined as

$D^{\gamma}[f(t)] = \frac{1}{\Gamma(n-\gamma)}\int_0^t (t-s)^{n-\gamma-1} f^{(n)} ds$ for n – 1 < γ ≤ n, ∀ n ∈ N, t > 0

**Definition 6** Laplace Transform of fractional derivatives.

The Laplace Transform g $D^{\gamma}[f(t)]$ is,

$$L[D^{\gamma}[f(t)] = S^{\gamma}L[f(t)] - S^{\gamma-1}f(0)S^{\gamma-2}f(0)$$

$$= S^{\gamma}L[f(t)] - \sum_{k=0}^{n-1} S^{\gamma-k-1} f^{k}(0) \text{ for } n-1 < \gamma \le n \ldots. \quad (3)$$

**Definition 7** The Riemann – Liouville fractional integral operator of order γ > 0 of a function f ∈ Cμ, μ ≥ 1, is defined as, where Cμ is the space, μ ∈ R.

$$J^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s)ds, \gamma > 0$$

$$J^0 f(t) = f(t)$$

### 7.1 Properties of Operator $J^{\gamma}$

For f ∈ Cμ, μ ≥ -1, β ≥ 0 and β ≥ - 1.

$J^{\gamma} J^{\beta} f(t) = J^{\gamma+\beta} f(t)$

$J^{\delta} J^{\mu} f(t) = J^{\beta} J^{\gamma} f(t)$

$$J^{\gamma} t^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\gamma+\delta+1)} t^{\gamma+\delta}$$

## III. PROCEDURE DESCRIPTION OR PROCEDURE ANALYSIS

*3.1 Procedure for Volterra Integral equation solution.*

Consider Volterra fractional integro-differential equation.

$$D^{\gamma} y(t) = r(t)y(t) + g(t) + \lambda\int_0^t k(t,s) f[y(s)]ds \quad (1)$$

First, operating Laplace Transform on both sides of (1), we get,

$$L[D^{\gamma} y(t)] = L[r(t)y(t)] + L[g(t)] + L\left[\lambda\int_0^t k(t,s) f[y(s)]ds\right]$$

Using (3) definition (6)

$$S^{\gamma}F(s) - \sum_{k=0}^{m-1} s^{\gamma-k-1} y^{(k)}(0) = L[r(t)y(t)] + L[g(t)]$$

$$+ L\left[\lambda\int_0^t k(t,s) f[y(s)]ds\right]$$

$$\Rightarrow F(s) = L[f(t)] - \frac{\sum S^{\gamma-k-1} y(k)(0)}{S^{\gamma}} + \frac{1}{S^{\gamma}} L[r(t)y(t)]$$

$$+ \frac{1}{S^{\gamma}} L[g(t)] + \frac{1}{S^{\gamma}} L\left[\lambda\int_0^t k(t,s) f[y(s)]ds\right]$$

$$L[f(t)] - \frac{b}{S^{\gamma}} + \frac{1}{S^{\gamma}} L[r(t)y(t)] + \frac{1}{S^{\gamma}} L[g(t)]$$

$$+ \frac{1}{S^{\gamma}} L\left[\lambda\int_0^t k(t,s) f[y(s)]ds\right] \quad (4)$$

Under Laplace decomposition method, the second step is that we represent solution as an infinite series give below,

$$y(t) = \sum_{n=0}^{\infty} y_n \quad (5)$$

The non-linear operator is decomposed as

$$Ny = F[y(t) = \sum_{n=0}^{\infty} Pn\ (y) \quad (6)$$

Where

Pn is the Adomian Polynomials for non-linear operators, of $y_0, y_1, y_2 \ldots. y_n$ … that are give by

$$P_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{t=0}^{\infty}\lambda^i y_i\right)\right] \quad n = 0, 1, 2...$$

For the non-linear function Ny = F(y) the first Adomian Polynomials are given by,

$$\left.\begin{aligned}
P_0 &= F(y_0) \\
P_1 &= y_1 F^{(1)}(y_0)\frac{1}{2!} \\
P_2 &= y_2 F^{(1)}(y_0) + \frac{1}{\alpha!} y_1^2 F^{(2)}(y_0) \\
P_3 &= y_3 F^{(1)}(y_0) + y_1 y_2 F^{(2)}(y_0) + \frac{1}{3!} y_1^3 F^{(3)}(y_0) \\
P_n &= \sum_{u=1}^{n} b(u,n) F^{(u)}(y_0)
\end{aligned}\right\} \quad (7)$$

The first index of b(u, n) is the order of derivatives from 1 to n and the second is of the order or the Adomian polynomial. The b(u, n) are products (or sums of products) of Cu components of whose subscripts sum to n, divided by the fractional of the no of repeated subscripts.

Subsituting (5) and (6) into (7) we will get

$$L\left[\sum_{n=0}^{\infty} y_n\right] = \frac{b}{S^{\gamma}} + \frac{1}{S^{\gamma}} L[g(t)] + \frac{1}{S^{\gamma}} L\left[p(t)\sum_{n=0}^{\infty} y_n\right]$$

$$+ \frac{\lambda}{S^{\gamma}} L\left[\int_0^t k(t,s)\sum_{n=0}^{\infty} F_n(y)ds\right] \quad (8)$$

Comparing both sides of (7) and (8) yields the following algorithim of each every interation.

$$L\left[y_0\right] = \frac{b}{S^{\gamma}} + \frac{1}{S^{\gamma}} L[y(t)] \quad (9)$$

$$L\left[y_1\right] = \frac{1}{S^{\gamma}} L[P(t)y_0] + \frac{1}{S^{\gamma}} L\left[\lambda \int_0^t k(t,s) F_0(y) ds\right] \quad (10)$$

$$L\left[y_2\right] = \frac{1}{S^{\gamma}} L[P(t)y_1] + \frac{1}{S^{\gamma}} L\left[\lambda \int_0^t k(t,s) F_1(y) ds\right] \quad (11)$$

In general, the recursive relation is given by,

$$L\left[y_{n+1}\right] = \frac{1}{S^{\gamma}} L[P(t)y_0] + \frac{1}{S^{\gamma}} L\left[\lambda \int_0^t k(t,s) F_n(y) ds\right] \quad (12)$$

Operating Laplace inverse transform on both sides of (9), (10), (11), (12) we will get recussive relation as follows.,

$$y_0(t) = R(t) \quad (13)$$

and $y_{n+1}(t) = L^{-1}\left[\frac{1}{S^{\gamma}} L[P(t)y_0]\right] + L^{-1}\left[\frac{1}{S^{\gamma}} L\left\{\lambda \int_0^t k(t,s) F_n(y) ds\right\}\right]$ (14)

Where R(t) is the function that arises from the source term and prescribed initial condition. The modified Laplace decomposition method, whereas is discussed by Y.Khen [14] suggests that the function R(t) defined above isi (13) be decomposed into two parts.

$$R_1(t) = R_1(t) + R_2(t)$$

We write the new modification results

$$y0(t) = R1(t) \quad (15)$$

$$y_1(t) = R_2(t) + L^{-1} \frac{1}{S^{\gamma}} L\left[\frac{1}{S^{\gamma}} L[P(t)y_0]\right] + L^{-1}\left[\lambda \int_0^t k(t,s) F_0(y) ds\right] \quad (16)$$

$$y_{n+1}(t) = L^{-1}\left[\frac{1}{S^{\gamma}} L\left\{P(t)y_n\right\}\right] + L^{-1}\left[\frac{1}{S^{\gamma}} L\left\{\lambda \int_0^t k(t,s) F_n(y) ds\right\}\right] \quad (17)$$

(15) – (17) are the solutions under modified Laplace decomposition method.

## IV. NUMERICAL EXAMPLES

Consider the linear fractional Volterra integro-differential equation which was considered by E.A. Rawashdeh [8].

$$D^{3/4} y(t) = \frac{6t}{\Gamma\frac{13}{4}} + \left(\frac{-t^2 e^t}{5}\right) y(t) + \int_0^t e^t\, s y(s) ds \quad (18)$$

With the initial condition, y(0) = 0 and the exact solution is $y(t) = t^3$.

*Solution*:

Operating Laplace Transform on both sides of (18),

$$L\left[D^{3/4} y(t)\right] = L\left[\frac{6t^{9/4}}{\Gamma\frac{13}{4}}\right] + L\left(\frac{-t^2 e^t}{5}\right) y(t) + L\left[\int_0^t e^t\, s y(s) ds\right]$$

$$= \frac{6}{\Gamma\frac{13}{4}} L(t^{9/4}) + \frac{1}{5} L\left[\left(-e^t t^2\right) y(t)\right] + L\left[\int_0^t e^t s y(s) ds\right]$$

$$= \frac{6}{\Gamma\left(\frac{13}{4}\right)}\left[\frac{\frac{9}{4}}{S^{\frac{13}{4}}}\right] - \frac{1}{5}\left[\left(\frac{2!}{S^3}\right)_{s\to s-1} y(t)\right] + L\left[\int_0^t e^t s y(s) ds\right]$$

$$L\left[D^{3/4} y(t)\right] = \frac{6\left(\frac{9}{4}\right)!}{\Gamma\left(\frac{13}{4}\right) S^{\frac{13}{4}}} - \frac{1}{5}\left[\left(\frac{2!}{(S-1)^3}\right) y(t)\right] + L\left[\int_0^t e^t s y(s) ds\right]$$

Using one of the properities of Laplace Transform and applying the initial condition y(0) = 0,

$$L\left[\sum_{n=0}^{\infty} y_n\right] = \frac{1}{S^{3/4}}\left\{\frac{6\left(\frac{9}{4}\right)!}{S^{13/4}\Gamma\left(\frac{13}{4}\right)}\right\} + \frac{1}{S^{3/4}} L\left[\frac{1}{5}\left\{\frac{2!}{(S-1)^3}\right\} y(t)\right] + \frac{1}{S^{3/4}} L\left[\int_0^t e^t s \sum_{n=0}^{\infty} y_n(s) ds\right] \quad (19)$$

Equating like co-efficient of (19) we have the following relation.

$$L\left[y_0\right] = \frac{1}{S^{3/4}} L\left\{\frac{6t^{9/4}}{\Gamma\left(\frac{13}{4}\right)}\right\}$$

$$L\left[y_1\right] = \frac{1}{S^{3/4}} L\left[\left(\frac{-t^2 e^t}{5}\right) y_0\right] + \frac{1}{S^{3/4}} L\left[\int_0^t e^t s\, y_o(s) ds\right]$$

$$L\left[y_{n+1}\right] = \frac{1}{S^{3/4}} L\left[\left(\frac{-t^2 e^t}{5}\right) y_n\right] + \frac{1}{S^{3/4}} L\left[\int_0^t e^t s\, y_n(s) ds\right] \quad (20)$$

Operating Laplace inverse transform on both sides of (20), we get,

$$y_0 = t^3$$

$$y_1 = L^{-1}\left[\frac{1}{S^{3/4}} L\left(\frac{-t^2 e^t}{5}\right) t^3\right] + L^{-1}\left[\frac{1}{S^{3/4}}\left\{L\int_0^t e^t (s) S^3\right\}\right] = 0$$

$y_{n+1} = 0$

Hence the solution is obtained ap

$$y(t) = \sum_{n=0}^{\infty} y_n = t^3$$

## V. CONCLUSION

In this paper, we applied the modified Laplace decomposition method to finding the approximate solution of linear Volterra fractional INtegro-differential equation by using Adomian Polynomials. It provides more realistic series solutions that coverage very rapidly in real physical problems. Lastly, the behavior of the solution can be formally determined by using basic Laplace Transform technique.

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