Optimization of a Fuzzy Matrix Game Under Dominance Approach

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Abstract - In this paper we present a dominance approach to solve the Fuzzy Game using LU type, Trapezoidal fuzzy number instead of crisp number. The solution of such Fuzzy Game with pure strategies by using minimax principle.

Keywords: Fuzzy Games, dominance property, imprecise numbers, LU type Trapezoidal Fuzzy Number.

I. INTRODUCTION

Game theory is the study of a collection of mathematical model of a conflict and cooperate between intelligent individual decision makers the term game theory goes well beyond recreation activities like prior games. The theory in its true sense deals with the ability of one entity or individual to take a certain decision keeping in view the effect of other entities decisions on him , in a situation of confident .The progress of game theory continued since its inception and used in many offer fields other than economics .Any game ,when played consists of participants called players or agents of the game ,each having his own preference or goal . Each player of the game has an associated amount of benefit or gain ,which be received at the end of the game called pay-off or utility , which measure the degree of stratification on individual player derives from the conflicting situation , for each player of the game ,the choice s available to the items are strategies .Game theory has now become an important mathematical tool which is used in situations that involves several entities whose decisions are influenced by the decisions of other entities playing with them.

The concept of modern game theory was introduced by Jon Von Newmann and Osker Morgenstern 1944 ,who described the ward 'Game ' for the first time by systematically specifying the rules of the game , the move of players information they posses during their moves and the outcome for each player at the end of the game .

However in real life situations the information available is of imprecise nature and there is an inherent degree of vagueness or uncertainty present in the system under consideration .In order to handle this uncertainty the concept of fuzzy sets can be used as an important decision making tool.

In this article ,we have concentrated on the solution of Two Person Zero- Sum Games by using LU- type rectangular fuzzy numbers they are characterized by their simple formations and computational efficiency and then have been used extensively to solve different problems in engineering and supply chain management .the solution of Fuzzy Game under 2x2 or 3x3 matrix.

II. TYPES OF FUZZY MEMBERSHIP FUNCTIONS

2.1 Triangular membership function

A Triangular membership function of a vector x and on three scalar parameters a ,b, c as given by

$$trim\ f(x,a,b,c) = \begin{cases} 0 & ,x \leq a \\ \frac{x-a}{b-a} & ,a \leq x \leq b \\ \frac{c-x}{c-b} & b \leq x \leq c \\ 0 & ,c \leq x \end{cases}$$

or more compactly by

$$\text{trimf}(x,a,b,c) = \max\left(min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right).$$

2.2 Trapezoidel membership function

A Trapezoidal membership function is defined by four parameters viz. a,b,c,d as follows

$$\text{Trapezoid } f(x,a,b,c,d) = \begin{cases} 0 & ,x \leq a \\ \frac{x-a}{b-a} & ,a \leq x \leq b \\ 1 & ,b \leq x \leq c \\ \frac{d-x}{d-c} & ,c \leq x \leq d \\ 0 & ,d \leq d \end{cases}$$

2.3 Gaussian membership function:

The gaussian membership function depends on two parameters σ and c as given by

$$\text{Gaussmf}(x: \sigma ,c)) = e^{\frac{-(x-c)^2}{2a^2}}$$

2.4 S- shaped membership function :

The spline – based curve is a mapping on ,x vector x and is named because of its S-shape .This membership function defined on two parameter a and b locate the extremes of the sloped partition of the curve are given by

$$\text{Smf}\ (x, a,b) = \begin{cases} 0 ; & x \le a \\ 2(\frac{x-a}{b-a})^2 ; & a \le x \le \frac{a+b}{2} \\ 1-2(\frac{x-a}{b-a})^2 ; & \frac{a+b}{2} \le x \le b \\ 1 ; & x \ge b \end{cases}$$

III. INTERVAL NUMBERS:

An interval number [4] is defined as

$A =[L_X,U_X] =\{ x : L_X \le x \le U_X , x \in R\}$.

Anything of representing an interval number in term of midpoint is A = <m(i),w(i)> where $m(i)=\frac{L_X+U_X}{2}$ and (i) = half width of A = $\frac{U_X-L_X}{2}$. Addition of two interval numbers $A = [L_X,U_X]$ and $B= [L_y,U_y]$ is $A+B =[L_x+ L_y , U_x+ U_y]$, using mean width notations ,if $A = <m_1,w_1)>$ and $B = <m_2,w_2>$ then $A+B = <m_1 + m_2 , w_1 +w_2>$ .Similarly the other binary operations on interval numbers are defined[4].

3.4 Ordered Relation Among Intervals:

If A = [a,b] ,B= [c,d] then [a,b] < [c,d] iff b < c and is denoted by A < B .Again A is contained in B iff a ≥c , b≥ d and is denoted by A≥B.

Definition 1 :

The dominance index (DI) to proposition A is dominated over B as A.

ie . $DI(A<B) = (m_2 - m_1)/(w_1+w_2)$ .Using DI is the following order is defined.

Definition 2:

If DI(A<B) ≥1 then A is said to be totally dominating over B in the sense minimization and B is said to be totally domination over A in the sense of minimization . This is denoted by A<B.

Definition 3:

If DI(A<B) < 1 then A is said to be partially dominating over B in the sense minimization

and B is said to be partially domination over A in the sense of minimization . This is denoted by A< B.

In special case:

When DI(A<B) < 1 , then $m_1=m_2$ it may be emphasized on the width of interval numbers A and B .If $w_1<w_2$,then the lower bound of A is less than that of B and there is a chance that on finding a minimum distance may be on A . But at the same time since the upper bound of A is greater than that of B ,if one prefers A over B in the minimization then in worst case he may be Looser than one who prefer A over B.

Numerical example:

consider A=[130,140] = < 135,5> &

B=[150,160]=<155,5>. The dominated index DI(A<B) =<155-135>/10=2 >1.So in minimization A is totally dominated over B.

Definition 4:

The Dominated index DI of proposition P =(α,a,δ) ,is dominated over Q= (β,b,δ) is given by DI(P<Q) = (b-a)/($\gamma+\delta$) .Using DI the following ranking order is defined.

Definition 5:

If DI(P<Q) ≥1 then P is said to be totally dominating over Q in the sense minimization

and Q is said to be totally domination over P in the sense of maximization ,it is also denoted by P<Q.

Definition 6:

If 0 < DI(P<Q) < 1 then P is said to be partially dominating over Q in the sense minimization and Q is said to be partially domination over P in the sense of maximization . It is also denoted by A<B.

Lemma 1:

If DI(P<Q) =0 then P and Q are said to be non comparable and is denoted by P≠Q .In this case P is preferred over Q if (a) $\alpha = \beta$ and $\gamma < \delta$ or (b) $\alpha > \beta$ and $\gamma = \delta$. Other wise a pessimistic decision maker would prefer the number with smaller length of support where as an optimistic decision maker would do the converse.

IV. TWO PERSON ZERO- SUM GAMES AND PAY – OFF MATRIX.

In this section we give some basic definitions of the Two Person Zero- sum games and pay off Matrix. From the basic building blocks of Game Theory .

4.1 Two Person Zero - Sum Games:

A game of two persons in which gains of one player is called a Two Person Zero -sum Game, i.e.,in Two Person Zero- Sum Game the algebraic sum of gains to both players after a play is to be zero.

Games relating to pure strategies taken by players are considered here based two assumptions [1,2] .

1. Player A is better position and is called maximization player (or row player) and Player B is called minimizing player (or column player).
2. Total gain of one player is exactly equal to total loss of other player .In general , if player A Takes m pure strategies and B take n pure strategies , then the game is called two person zero sum game or m x n rectangular game.

4.2 Two person constant Games:

Add the matrix payoff player 1 and matrix payoff 2 if the resulting matrix has all entries equal the is a constant sum game.

Example 1 : (Constant sum game)

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 0.2 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 \\ 1 & 0.8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Pay-off of player1 *pay-off of player2* *pay-off sum game*

Example 2: (not constant sum game)

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 0.2 \end{bmatrix} + \begin{bmatrix} 1 & 0.5 \\ 1 & 0.8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Pay-off of player1 *pay-off of player2* *pay-off sum game*

4.3 Pay-off matrix :

Two Person Zero - Sum Games are known as rectangular games since they are represented by Rectangular pay – off matrix . A pay-off matrix is always written for maximizing player. Considering the general m x n rectangular game ,the pay-off matrix of A with m pure strategies A_1, A_2 , A_3……………..A_m. And B with n pure strategies B_1,B_2,B_3,……….......B_n is given by [3].

$$\begin{bmatrix} < a_{11}, b_{11} > & \dots\dots\dots.. & < a_{1n}, b_{1n} > \\ \dots\dots\dots\dots\dots & \dots\dots\dots\dots & \dots\dots\dots\dots.. \\ \dots\dots\dots\dots\dots & \dots\dots\dots\dots & \dots\dots\dots\dots. \\ \dots\dots\dots\dots\dots & \dots\dots\dots\dots & \dots\dots\dots\dots\dots \\ < a_{m1}, b_{m1} > & \dots\dots\dots\dots & < a_{mn}, b_{mn} > \end{bmatrix}$$

The elements $<a_{ij}, b_{ij}>$ are LU –Type trapezoidal fuzzy numbers and for crisp game they may be positive, negative, or zero. When player A chooses strategy A_i and player B selects B_j , it results in pay- off LU-type trapezoidal fuzzy number $<a_{ij},b_{ij}>$ to player A.

V. SOLUTION OF 2X2 GAMES WITH MIXED STRATEGIES

Consider the fuzzy game [3,5] of players A (strategies represented by horizontally) and B (strategies represented by vertically) whose pay-off is given following matrix and for which there is no saddle point.

$$\begin{bmatrix} < a_{11}, b_{11} > & < a_{12}, b_{12} > \\ < a_{21}, b_{21} > & < a_{22}, b_{22} > \end{bmatrix}$$

Where ,pay-off <aij,bij > are symmetric LU –type trapezoidal fuzzy number such that bij = aij +α .If

X_i and y_j be probabilities by which A chooses i^{th} strategy and B chooses j^{th} strategy then :

$$X_1 = (a_{22}-a_{21})/(a_{11}+a_{22} - a_{12} - a_{21}) ;$$

$$Y_1 = (a_{22}-a_{21})/(a_{11}+a_{22} - a_{12} - a_{21}) :$$

$$X_2 = (a_{11}-a_{12})/(a_{11}+a_{22} - a_{12} - a_{21}) ;$$

$$Y_2 = (a_{11}-a_{12})/(a_{11}+a_{22} - a_{12} - a_{21}) ;$$

Which are crisp numbers and value of the game can be easily computed as V= <a,b> ;where a and b are left and right spreads of LU -type trapezoidal fuzzy numbers given by ;

$$a = (a_{11}a_{22}-a_{12}a_{21})/(a_{11}+a_{22} - a_{12} - a_{21})$$

$$b = (b_{11}b_{22}-b_{12}b_{21})/(b_{11}+b_{22} - b_{12} - b_{21}) .$$

VI. GAMES WITH NO SADDLE POINT FOR M X N FUZZY GAME .

We consider m x n Fuzzy Game with no saddle point ,now we discuss a particular method, the pay- off can be reduced to 2x2 games so that it can be solved by using the Fuzzy Game method. The method of reduction of the pay-off matrix by this process is called the Dominance property of the rows and columns of the pay-off matrix.

VII. CONCEPT OF DOMINANCE PROPERTY

If one pure strategy of a player is better for him or as good as another, for all possible pure strategies of opponent then first is said to dominated the second [1,2] . The dominated strategy can simply be discarded from pay-off matrix since it has no value. When this is done, optimal strategies for the reduced matrix are also optimal for the original matrix with zero probability for discarded strategies. When there is no saddle point in pay-off matrix, then size of the game can be reduced by dominance, before the problem is solved.

Definition 7:

If all elements of the i^{th} row of pay-off matrix of a m x n rectangular game are dominating over r^{th} row in the sense of maximization, r^{th} row is discarded and deletion of r^{th} row from matrix does not change the set of optimal strategies of maximizing player.

Numerical Example:

Consider the fuzzy game of two players A (strategies represented horizontally) and B (strategies represented vertically) with the following pay-off matrix. Player A is maximizing player and player B is minimizing player.b_{11}

$$\begin{bmatrix} <2,0.2> & <8,0.3> & <2,0.1> \\ <7,0.2> & <3,0.1> & <8,0.3> \\ <1,0.2> & <2,0.2> & <7,0.2> \end{bmatrix}$$

$DI(A_{31}<A_{21})=(7-1)/(0.2+0.2) > 1$

$DI(A_{32}<A_{22})=(3-2)/(0.1+0.3) > 1$

$DI(A_{33}<A_{23})=(8-7)/(0.3+0.2) > 1$

Thus A_2 is dominating over A_3 in the sense of maximization and row A_3 is deleted . The reduced matrix is given by ,

$$\begin{bmatrix} <2,0.2> & <8,0.3> & <2,0.1> \\ <7,0.2> & <3,0.1> & <3,0.1> \end{bmatrix}$$

Definition 8: If all elements of j^{th} column are dominating over s^{th} column in the sense of minimization the s^{th} column is deleted and the deletion of s^{th} does not change the set of optimal strategies of minimizing player.

Numerical example :

Considering the above pay off matrix ,

$DI(B_{11}<B_{13})=(3-2)/(0.1+0.3) > 1$

$DI(B_{21}<B_{23})=(8-7)/(0.2+0.3) > 1$

Here B_1 is totally dominating over B_3 in the sense of minimization and the resultant pay-off matrix is given by

$$\begin{bmatrix} <2,0.2> & <8,0.3> \\ <7,0.2> & <3,0.1> \end{bmatrix}$$

Definition 9: If the linear combination p^{ih} row and q^{th} rows dominates all element of the s^{th} row in the sense of minimization, s^{th} row discard and the deletion of s^{th} row from matrix does not change the set of optimal strategies of maximizing player.

Numerical example:

Considering a particular pay-off matrix of two players A (strategies represented horizontally) and B (strategies vertically) as follows;

$$\begin{bmatrix} <3,0.4> & <4,0.1> & <1,0.1> \\ <5,0.5> & <3,0.5> & <4,0.2> \\ <1,0.2> & <5,0.4> & <4,0.4> \end{bmatrix}$$

The convex combination of second and third row gives $A_4 = \beta A_2 + (1-\beta)A_3$. $0\leq\beta\leq1$

Taking $\beta = 0.5$ the elements of A_4 are < 3,0.35>, <4,0.35> and <4,0.30>.Now A_4 is dominating over A_1 in the sense of maximization and row A_1 is discard such that the resulting pay-off matrix is given by.

$$\begin{bmatrix} <5,0.5> & <3,0.5> & <4,0.2> \\ <1,0.2> & <5,0.5> & <4,0.4> \end{bmatrix}$$

Definition 10 : If j^{th} column is dominated by the convex combination of m^{th} and n^{th} column,j^{th} column is discard in the sense of minimization and deletion of the j^{th} column from matrix does not change the set of optimal strategies of the minimizing player.

Numerical example : Considering above pay-off matrix the convex combination B_1 and B_2

i.e. $B_4 = \alpha B_1 + (1-\alpha)B_2$; $0 \leq \alpha \leq 1$. Taking $\alpha = 0.5$.

$$\begin{bmatrix} <4,0.5> \\ <3,0.3> \end{bmatrix}$$

Now B_4 is totally dominating over B_3 and thus the third column is removed such that the resulting matrix is given by

$$\begin{bmatrix} <5,0.5> & <4,0.4> \\ <1,0.2> & <3,0.3> \end{bmatrix}$$

VIII. SOLUTION FOR REDUCED 2XN OR MX2 PAYOFF MATRIX

Definition 11. When there is no saddle point and no course action dominates any other the values for different 2x2 sub games are computed .As A is maximizing player he will definitely select that pair strategies which will give the best value of 2x2 sub games and the corresponding sub matrix provides

Optimal solution. Similarly, B is minimizing player he will definitely select that pair of course, which will give the least of 2x2 sub games, the corresponding sub matrix will provide optimal solution to sub matrix will provide optimal solution to the Fuzzy Game.

Numerical example: consider the Fuzzy Game whose pay-off matrix is given by,

$$\begin{bmatrix} <21,0.2> & <17,0.4> & <18,0.1> \\ <2,0.2> & <22,0.4> & <7,0.4> \end{bmatrix}$$

There is no saddle point and no course of action dominates any other. The values v1,v2,v3 are computed from the following three 2x2 sub games as obtained from given matrix

Sub game 1

$$\begin{bmatrix} <21,0.2> & <17,0.4> \\ <2,0.2> & <22,0.4> \end{bmatrix}$$

The corresponding value $V_1 = <\frac{107}{6}, \frac{7}{5}>$

Sub game 2

$$\begin{bmatrix} <21,0.2> & <18,0.1> \\ <2,0.2> & <7,0.4> \end{bmatrix}$$

The corresponding value $V_2 = <\frac{111}{8}, 0.1>$

Sub game 3

$$\begin{bmatrix} <17,0.4> & <18,0.1> \\ <22,0.4> & <7,0.4> \end{bmatrix}$$

The corresponding value $V_3 = <\frac{83}{-16}, \frac{11}{5}>$

Here , min $\{V_1,V_2,V_3\}= V_3$ such the resulting pay- off matrix is sub game 3.

IX. NUMERICAL SIMULATION

To illustrate dominance method , we consider the following pay off matrix ;

$$\begin{bmatrix} <8,0.3> & <15,0.4> & <-4,0.1> & <-2,0.4> \\ <19,0.1> & <15,0.5> & <17,0.4> & <16,0.1> \\ <0,0.3> & <20,0.2> & <15,0.5> & <5,0.4> \end{bmatrix}$$

All $DI(A_1 < A_2) \geq 1$, so A_2 is totally dominating over A_1 in the sense of minimization and row A_1 is deleted, such that the resulting pay-off matrix is:

$$\begin{bmatrix} <19,0.1> & <15,0.5> & <17,0.4> & <16,0.1> \\ <0,0.3> & <20,0.2> & <15,0.5> & <5,0.4> \end{bmatrix}$$

Again all $DI(B_4 < B_3) \geq 1$, so B_4 is totally dominating over B_3 in the sense of minimization and the column B_3 is deleted such that the resulting pay-off matrix is:

$$\begin{bmatrix} <19,0.1> & <15,0.5> & <16,0.1> \\ <0,0.3> & <20,0.2> & <5,0.4> \end{bmatrix}$$

This is no course of action which dominates any other and there is no saddle point. The values for different 2x2 pair of strategies are computed. Since, B is minimizing player the minimum value is considered and corresponding pay-off matrix provides optimal solution to Fuzzy problem. The previous least value of $\{V_1, V_2,V_3\}$ is v_3 so optimal strategies of A are (A_2, A_3) and of B are (B_2,B_3). The final pay-off matrix is given by,

$$\begin{bmatrix} <15,0.5> & <16,0.1> \\ <20,0.2> & <5,0.4> \end{bmatrix}$$

The probabilities are

$x_1 = 0$, $x_2 = \frac{15}{16}$, $x_3 = \frac{1}{16}$ And $y_1 = 0$, $y_2 = \frac{1}{16}$, $y_3 = 0$, $y_4 = \frac{15}{16}$

And value of the game is $V = <\frac{245}{16}, \frac{11}{5}>$

Hence, optimal solution of the complete game is $(0,x_2,x_3)$ for A and $(0,y_2,0,y_4)$ for B; the value of game being V.

X. CONCLUSION

In this paper, we considered the solution of Rectangular Fuzzy Games using LU-type Trapezoidal Fuzzy Numbers. Here pay-off is considered as imprecise numbers instead of crisp numbers which takes care of the uncertainty and vagueness inherent in such problems. LU-type Trapezoidal Fuzzy Numbers are used because of their simplicity and computational efficiency. We discuss solution of Fuzzy Games with pure strategies by minimax-maximin principle and also Algebraic Method to solve 2X2 Fuzzy Games without saddle point by using mixed strategies. The Concept of Dominance Method is also illustrated. LU-type Trapezoidal Fuzzy Numbers generates optional solutions which are feasible in nature and also takes care of the impreciseness aspect.

XI. FUTURE WORK

The following few recommendation for further study

I. Most of the fuzzy matrix of the problem under dominance approach can be derive in different order m x n matrix, again it may extend various fuzzy number by using LU type problems.

II. Another specified extension of this approach it is to solve fuzzy game of big order m x n square matrix problem. Both in optimization mathematical finance and finance management areas.

III. The proposed approach can be assist in the game application areas such as traffic system and general distribution system in engineering problem.

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