Numerical Solution of Eighth Order Boundary Value Problems using Eleventh Degree Spline Functions

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Abstract: In this paper numerical solutions of general linear boundary value problem of order eight are considered. Eleventh degree spline approximations are developed following Cubic Spline Bickley's procedure and applied. Approximate numerical solutions are computed at varying step lengths. Approximate and exact solutions are compared. Also absolute errors are calculated. The results are tabulated and pictorially illustrated. Further, the results of the current method are compared with those of other popular ones.

Keywords: Spline functions, eighth order boundary value problems, Eleventh degree spline, Numerical results, two point boundary value problem.

1. INTRODUCTION

Higher order boundary value problems arise in the study of radiation transport in spherical and higher geometry with or without velocity field in astrophysics, stellar evolution, star formation and galaxy evolution. Initial – boundary value problems arise in modelling various areas and phenomena in fluid dynamics. The areas include predicting atmospheric processes and ocean circulation. The phenomena include convection, flow in wind tunnels, lee waves, and eddy. Free boundary problems do occur in fluid mechanics such as air flow around a wing of an aircraft, underground fluid flow through layers of soil and modelling of an ocean surface [1].

Higher order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamics, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified fields of applied sciences.

The behaviour of induction motors is modelled using fifth – order boundary value problems. This model contains five variables in all. Of them two each are related to stator and rotor states and the other one to shat speed. In fact, two more variables will be required to account for the effects of a second rotor circuit representing deep bars, a starting cage, or rotor distributed parameters. But, to avoid the computational burden of additional state variables when additional rotor circuits are required the model is often limited to the fifth – order and rotor impedance is algebraically altered as function of rotor speed under the assumption that the frequency of rotor currents depends on rotor speed [2].

The approach is efficient for the steady state response with sinusoidal voltage. But it does not hold well when the transient conditions are included i.e., rotor frequency is not a single value. So, the behaviours of such models are described using the seventh order boundary value problems [3].

Further, it is investigated that when an infinite horizontal layer of fluid is heated from below and is subject to rotation then the instability takes place [4]. If the instability sets in as over stability then the phenomenon is represented by an eighth – order boundary value problems.

If an infinite horizontal layer of fluid is heated from below with the assumption that a uniform magnetic field is applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation then the instability occurs. When this instability sets in as ordinary convection then it is modelled by tenth – order boundary value problems.

The method variation of parameters is used for solving the seventh – order problems [5]. Eighth – order problems are solved using various techniques including: general differential quadrature rule, Nonic spline and non polynomial spline technique, Octic spline, Adomian decomposition method, Homotopy perturbation method and Kernel space method [6-7]. Tenth – order boundary value problems are solved following few procedures including: reproducing Kernel method, variational iteration technique, numerical solutions and tenth degree spline [8-9].

A spline collocation method is developed using spline interpellants and analyzed the approximating solutions of some general linear boundary value problems [10]. The HAM and HPM are compared in solving non linear heat transfer equation [11]. HAM is employed to compute approximate solution of the system of differential equations governed by the problem and also is used to detect the in excellence of convective straight lines with temperature dependent thermal conductivity [12].

HAM is successfully applied to solve MHD Jefery – Hamel flows in nonparallel walls. Non polynomial spline is used in of – step points to solve special tenth order linear boundary value problems. Laplace decomposition method (LDM) is applied to nonlinear Blasius low equation to obtain series solutions. A new method has been constructed for finding the solution of Abel's type singular integral equations. However, Bikely's method makes the calculation much simpler [13].

The newly proposed method efficiently finds exact solution and can be used to solve nonlinear Volterra integral equations [14]. Coupling of Homotopy perturbation and Laplace transformation is proposed and is used for solving system of partial differential equations [15]. The stagnation point low of a viscous fluid towards a stretching sheet has been described and using which obtained an analytical solution of the boundary layer equation by HAM [16]. A modification of the HAM is available for solving nonlinear boundary value problems [17]. The accuracy of the HAM for solving the fractional order problem of the spread of a disease in a population has been investigated [18]. The application of HAM to general nonlinear Klein – Gordon type equations has been discussed [19].

Prior to 1950, the computations involved in numerical method to obtain numerical solution of a differential equation were done manually. Later these computations are carried through the calculators followed by digital computers. Due to rapid advances in computing machines like high speed and accuracy, several researchers have been working for developing the numerical methods to obtain the numerical solution of the ordinary or partial differential equations with specified conditions.

Initially, the cubic spline technique has been introduced for solving second order two point boundary value problems. A solution has been obtained for a set of linear equations whose coefficients form an upper Hessen berg matrix. The applications of cubic spline have been proved positive in the literature [20].

In this paper we have constructed eleventh degree spline function and applied to solve the linear eighth order differential equation. The numerical solutions are constructed for different step lengths. The results of the present method are compared with those of the other methods and important observations are drawn.

2. CONSTRUCTION OF ELEVENTH DEGREE SPLINE

Let the interval $[x_0,\ x_n]$ be divided into n subintervals with grid points at the locations $x_0,\ x_1, \ldots, x_n$. Starting at x_0 the function $y(x)$ in the interval $[x_0,\ x_1]$ is represented by eleventh degree spline as

$$f(x) = a + b(x-x_0) + c\,(x-x_0)^2 + d(x-x_0)^3 + e(x-x_0)^4 + g(x-x_0)^5 + h(x-x_0)^6 + i(x-x_0)^7 + j(x-x_0)^8 + k(x-x_0)^9 + l(x-x_0)^{10} + t(x-x_0)^{11} \quad (1)$$

Proceeding to the next interval $[x_1, x_2]$ we add a term $t_1(x-x_1)^{11}$, proceeding in to the next interval $[x_2,\ x_3]$ we add another term $t_2(x-x_1)^{11}$, and so on until we reach x_n. Thus the function $y(x)$ is represented in the form

$$f(x) = \mathrm{a} + \mathrm{b}(x-x_0) + \mathrm{c}(x-x_0)^2 + \mathrm{d}(x-x_0)^3 + \mathrm{e}(x-x_0)^4 + \mathrm{g}(x-x_0)^5 + \mathrm{h}(x-x_0)^6 + i(x-x_0)^7 + \mathrm{j}(x-x_0)^8 + k(x-x_0)^9 + l(x-x_0)^{10} + \sum_{i=0}^{n-1} t_i\left(x_n - x_0\right)^{11} \quad (2)$$

It can be shown that $f(x)$ and its first eighth derivatives are continuous across nodes.

2.1 METHOD OF OBTAINING THE SOLUTION OF EIGHTH ORDER BOUNDARY VALUE PROBLEMS USING ELEVENTH DEGREE SPLINE FUNCTION

Consider the linear seventh order differential equation together with boundary conditions

$$y^{(8)}(x) + f(x)y(x) = p(x)$$

$$\mathrm{y}(x^0) = \alpha,\ y(x^{\mathrm{n}}) = \beta,\ y'(x^0) = \alpha',\quad y'(x^{\mathrm{n}}) = \beta',\ y''(x^0) = \alpha'',\ y''(x^{\mathrm{n}}) = \beta'',\ y^{(3)}(x^0) = \alpha''',\ y^{(3)}(x_n) = \beta''' \quad (3)$$

From (3), and taking spline approximation in (10) at $x = x_i$ for $i = 0,1,2,3,4,\ldots,n$ we get $(n+11)$ equations in $(n+13)$ unknowns $a,b,c,d,e,g,h,i,j,k,l,\mathrm{t}_0,\ \mathrm{t}_1,\ \mathrm{t}_2,\ \mathrm{t}_3,\ldots,\mathrm{t}_{\mathrm{n}-1}$. To have the solution for the unknowns one more equation is required. So we assume that $t_{n-1} = \mathrm{t}_{\mathrm{n}-2} = \mathrm{t}_{\mathrm{n}-3}$ after determining these unknowns we substitute them in (2) and thus we get eleventh degree spline approximation of $y(x)$. Putting $x = x_1,\ x_2,\ x_3, \ldots,\ x_{\mathrm{n}}$ in the spline function thus determined we get the solution at the grid points. The system of equations to be satisfied by the coefficients $a,b,c,d,e,g,h,i,j,k,l,\mathrm{t}_0,\ \mathrm{t}_1,\ \mathrm{t}_2,\ \mathrm{t}_3,\ldots,\mathrm{t}_{\mathrm{n}-1}$ is derived below: From equation (2) we get

$$f^{(8)}(x) = 40320\,j + 362880\,k\,(x-x_0) + 1814400\,l\,(x-x_0)^2 + 6652800\sum_{i=0}^{n-1} t_i(x-x_i)^3 \quad (4)$$

Since $f(x)$ approximates $y(x)$, from (1) and from the boundary conditions (3) we obtain

$$a = \alpha , \tag{5}$$

$$a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + e(x - x_0)^4 + g(x - x_0)^5 + h(x - x_0)^6 + i(x - x_0)^7 + j(x - x_0)^8 + k(x - x_0)^9 + l(x - x_0)^{10} + \sum_{i=0}^{n-1} t_i \left(x_n - x_0\right)^{11} = \beta \quad (6)$$

$$b = \alpha', \tag{7}$$

$$b + 2c\left(x - x_0\right) + 3d\left(x - x_0\right)^2 + 4e\left(x - x_0\right)^3 + 5g\left(x - x_0\right)^4 + 6h\left(x - x_0\right)^5 + 7i\left(x - x_0\right)^6 + 8j\left(x - x_0\right)^7 + 9\,k(x - x_0)^8 + 10l\,(x - x_0)^9 + 11\sum_{i=0}^{n-1} t_i \left(x - x_i\right)^{10} = \beta' (8)$$

$$2c = \alpha'' \tag{9}$$

$$2c + 6d\left(x - x_0\right) + 12e\left(x - x_0\right)^2 + 20g\left(x - x_0\right)^3 + 30h\left(x - x_0\right)^4 + 42i\left(x - x_0\right)^5 + 56j\left(x - x_0\right)^6 + 72\,k(x -$$
x07+90l x−x08+110i=0n−1tix−xi9=β′′ (10)

$$6d = \alpha''' \tag{11}$$

$$6d + 24e\left(x - x_0\right) + 60g\left(x - x_0\right)^2 + 120h\left(x - x_0\right)^3 + 210i\left(x - x_0\right)^4 + 336j\left(x - x_0\right)^5 + 504\,k(x - x_0)^6 + 720l\,(x - x_0)^7 + 990\sum_{i=0}^{n-1} t_i \left(x - x_i\right)^8 = \beta''' \quad (12)$$

From $(5) - (12)$ we have $(n + 11)$ equations, if these equations are taken in the order (7), (9), and (11) with $m = n, n - 1, \ldots, 0$, (12), (10), (8) and (6) the coefficient matrix of the unknowns, $t_n,\ t_{n-1},\ \ldots t_1,\ t_0,\ l,\ k,\ j,\ i,\ h,\ g,\ e,\ d,\ c,\ b,\ a$ will be an upper triangular matrix with two lower sub diagonals. The forward elimination is then simple with only two multipliers at each step, and back substitution is correspondingly made easy using MATLAB software.

3. NUMERICAL RESULT

In this section we consider one linear non homogeneous boundary value problem. Its numerical solution and absolute errors are given at different step lengths. The approximate solution, exact solutions and absolute errors at the grid points are summarized in tabular form. Further the approximate solution and exact solution have been shown graphically. The comparison of maximum absolute errors at different step lengths has been presented in tabular form.

Problem

Consider the following homogeneous linear eighth order boundary value problem

$$u^{(8)}(x) = -x\,u(x) - e^x\,(48 + 15x + 2x^3), 0 \le x \le 1 \tag{13}$$

With boundary conditions

$u(0) = 0,\ u'(0) = 1,\ u^{(2)}(0) = 0,\ u^{(3)}(0) = -3,\ u(1) = 0,\ u'(1) = -e,\ u^{(2)}(1) = -4e,\ u^{(3)}(1) = -9e$
(14)

The exact solution is $u(x) = x(1 - x)e^x$.

We find the solution of (13) – (14) by taking step lengths $h = 0.2$ and $h = 0.1$ at equal subintervals.

Solution with $h = 0.2$

The eleventh degree spline $f(x)$ which approximates $u(x)$ is given by

$$f(x) = \mathrm{a} + \mathrm{b}(x - x_0) + \mathrm{c}(x - x_0)^2 + \mathrm{d}(x - x_0)^3 + \mathrm{e}(x - x_0)^4 + \mathrm{g}(x - x_0)^5 + \mathrm{h}(x - x_0)^6 + i(x - x_0)^7 + \mathrm{j}(x - x_0)^8 + k(x - x_0)^9 + l(x - x_0)^{10} + \sum_{i=0}^{n-1} t_i \left(x_n - x_0\right)^{11} \quad (15)$$

Here in (15), $x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,\ x_5 = 1$ and we have 17 unknowns viz., $a,\ b,\ c,\ d,\ e,\ g,\ h,\ i,\ j,\ k,\ l,\ t_0,\ t_1,\ t_2,\ t_3,\ t_4$ and the conditions to be satisfied by these unknowns are

$f(x_0) = 0,\ f(x_5) = 0,\ \ f'(x_0) = 1,\ f'(x_5) = -e,\ \ f^{(2)}(x_0) = 0,\ \ f^{(2)}(x_5) = -4e,\ f^{(3)}(x_0) = -3 ,\ f^{(3)}(x_5) = -9e$ (16)

$$f^{(8)}(x_i) = -xf(x_i) - e^{x_i}\left(48 + 15x_i + 2{x_i}^3\right), \text{for } \ i = 0, 1, 2, 3, 4. \tag{17}$$

Since $f(x_0) = 0,\ f'(x_0) = 1, f''(x_0) = 0,\ f^{(3)}(x_0) = -3,$ it follows that $a = 0,\ b = 1,\ c = 0$ and $d = -0.5$ hence the spline $S(x)$ reduces to the form

$$f(x) = (x - x_0) - 0.5(x - x_0)^3 + e(x - x_0)^4 + g(x - x_0)^5 + h(x - x_0)^6 + i(x - x_0)^7 + j(x - x_0)^8 + k(x - x_0)^9 + l(x - x_0)^{10} + \sum_{i=0}^{4} t_i\,(x_n - x_0)^{11} \quad (18)$$

Solving (17) we have the following values for the unknown coefficients

$e = -0.3297$ $\quad k = 0.001440$ $\quad t_3 = 0.000398$

$g = -0.134280$ $\quad l = -0.002809$ $\quad t_4 = 0.000398$

$h = -0.025108$ $\quad t_0 = 0.001597$

$i = -0.009710$ $\quad t_1 = -0.002002$

$j = -0.001190$ $\quad t_2 = 0.000398$

Substituting these values in equation (15) we get the spline approximation $f(x)$ of $u(x)$.

The values of $f(x)$, $u(x)$ and the corresponding absolute errors at x_1, x_2, x_3, x_4 have been given in the Table 1 and the comparison has been shown in Figure 1.

TABLE 1: Numerical solution $f(x)$, exact solution $u(x)$ and absolute error of the problem with $h = 0.2$

x	$f(x)$	$u(x)$	Absolute error
0.2	0.195424	0.195424	3.7440E – 09
0.4	0.358037	0.358037	3.3611E-09
0.6	0.437308	0.437308	7.5699E-08
0.8	0.356086	0.356086	6.0345E-08

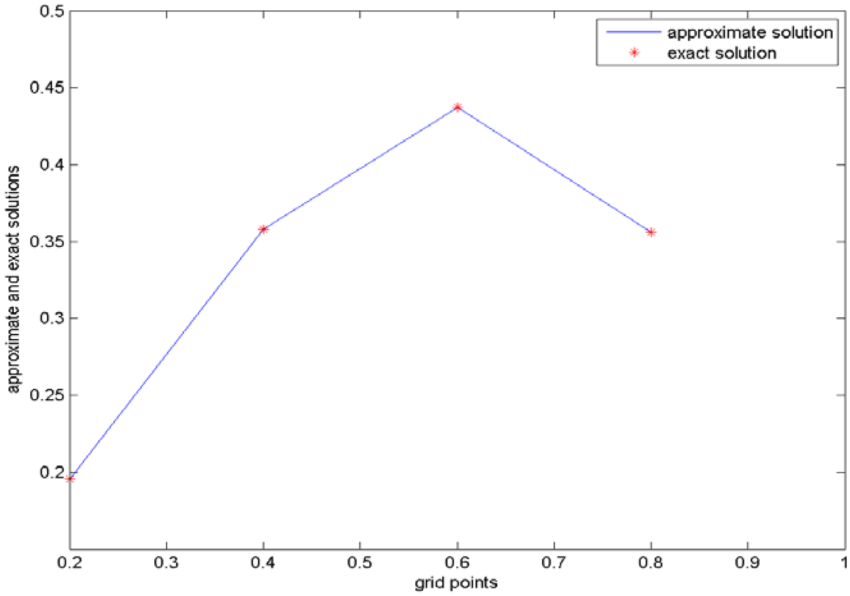

Figure 1 Comparison of exact and approximate solutions of the problem with $h = 0.2$ Solution with $h = 0.1$

Since $h = 0.1$ we suppose the grid points $x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5, x_6, x_7, x_8, x_9, x_{10}$ where, $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4, x_5 = 0.5, x_6 = 0.6, x_7 = 0.7, x_8 = 0.8, x_9 = 0.9, x_{10} = 1.0$ From equation (1) eleventh degree spline $f(x)$ which approximates $u(x)$ becomes

$$f(x) = (x - x_0) - 0.5(x - x_0)^3 + e(x - x_0)^4 + g(x - x_0)^5 + h(x - x_0)^6 + i(x - x_0)^7 + j(x - x_0)^8 + k(x - x_0)^9 + l(x - x_0)^{10} + \sum_{i=0}^{9} t_i \ (x_n - x_0)^{11} \quad (19)$$

From equation (18) and the boundary conditions we get the following values: $e = -0.2215407$, $g = -0.0144194$, $h = 0.00160134$, $i = -0.009710$, $j = -0.001190$, $k = -0.0002199$, $l = -0.0023378$, $t_0 = 0.0042809$, $t_1 = -0.00433413$, $t_2 = 0.004274$, $t_3 = -0.004341$, $t_4 = 0.0042651$, $t_5 = -0.0043519$, $t_6 = 0.0034716$, $t_7 = 0.0034716$, $t_8 = 0.0034716$, $t_9 = 0.0034716$

Substituting these values in equation (18) we get the spline approximation $f(x)$ of $u(x)$. The values of $f(x)$, $u(x)$ and the corresponding absolute errors at x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} have been given in the Table 2 and the comparison has been shown in Fig 2.

TABLE 2 NUMERICAL SOLUTIONS $f(x)$, EXACT SOLUTION $u(x)$ AND ABSOLUTE ERROR OF THE PROBLEM WITH $h = 0.1$

x	$f(x)$	$u(x)$	Absolute error
0.1	0.099465	0.099465	3.9968E-15
0.2	0.195424	0.195424	3.6801E-15
0.3	0.283470	0.283470	1.3001E-15
0.4	0.358037	0.358037	7.3580E-14
0.5	0.412180	0.412180	1.0000E-14
0.6	0.437308	0.437308	5.7009E-14
0.7	0.428881	0.428880	1.0003E-13
0.8	0.356086	0.356086	2.0000E-13
0.9	0.221364	0.221364	2.0400E-13

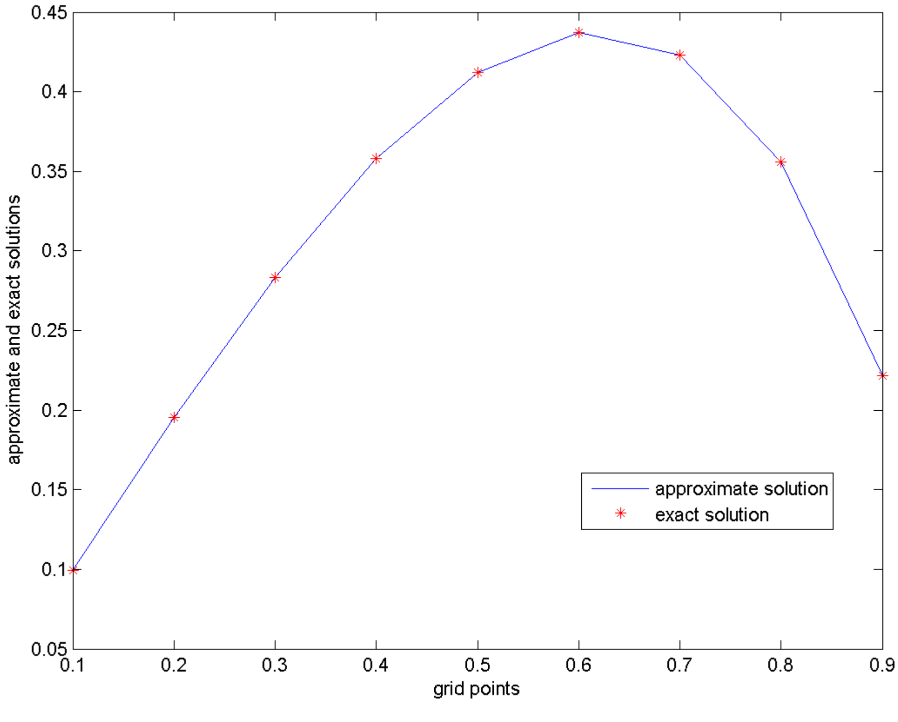

Figure 2. Comparison of approximate and exact solution of the problem with $h = 0.1$

4. COMPARATIVE STUDY OF PRESENT METHOD WITH OTHER METHODS

The numerical result obtained by present method is compared with the numerical result obtained by other methods.

TABLE 3: COMPARISON OF THE ABSOLUTE ERRORS OBTAINED USING DIFFERENT METHODS FOR THE PROBLEM AT $h = 0.1$

x	Exact Solution	Approximate Solution	Absolute error of				
			Present Method	HAM Method [20]	Akram and Rehman [6]	Siddiqi & Akram [4]	Inc and Evans [5]
0.1	0.099465	0.099465	3.9968E-15	3.89966E-15	1.63E-10	5.62E-10	3.73E-09
0.2	0.195424	0.195424	3.6801E-15	9.45355E-14	1.63E-09	4.88E-09	6.62E-09
0.3	0.283470	0.283470	1.3001E-15	7.04437E-14	4.90E-09	1.37E-08	2.33E-08
0.4	0.358037	0.358037	7.3580E-14	4.36873E-13	8.46E-09	2.29E-08	5.17E-08
0.5	0.412180	0.412180	1.0000E-14	1.28897E-13	1.01E-08	2.71E-08	9.76E-08
0.6	0.437308	0.437308	5.7009E-14	4.01956E-13	8.68E-09	2.38E-08	1.76E-06
0.7	0.428880	0.428880	1.0003E-13	2.06929E-12	5.15E-09	1.49E-08	4.12E-06
0.8	0.221364	0.221364	2.0000E-13	2.65915E-12	1.76E-09	5.54E-09	1.83E-04

From the table 3 above the present method is more accurate than the other methods which is shown graphically as follows:

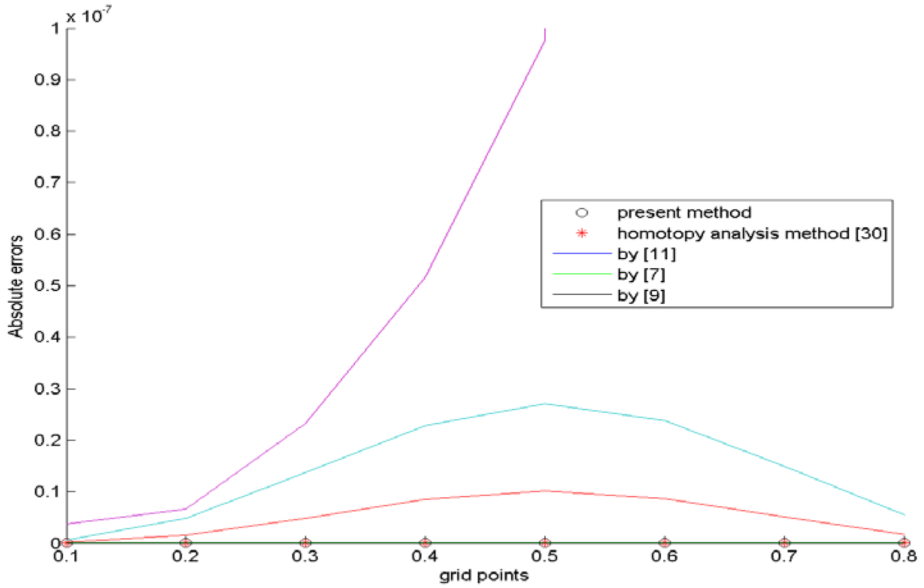

Figure 3: Comparison of absolute errors obtained using different methods for the problem 1 with $h = 0.1$

5. CONCLUSIONS

In this paper we developed the numerical methods to obtain the solution of eighth order boundary value problems using eleventh degree spline. Eleventh degree spline approximation has been employed on one problem at different step lengths. Numerical solution of the problem has been found with $h = 0.2$ and $h = 0.1$. Approximate solution, exact solution and absolute errors with $h = 0.2$ and $h = 0.1$ of the problem are summarized in the Table 1 and Table 2 respectively. The comparison has been shown in Figure 1 and 2 respectively.

The minimum absolute errors or the maximum accuracy at these step length are 3.7440×10^{-9}, and 3.9968×10^{-15} respectively. From this we understand that there is good agreement with the exact solution. It is also observed that the approximate solution is more close to the exact solution when h is small.

In this section, the spline approximation method has been applied to obtain the numerical solutions of eighth order boundary value problems using eleventh degree spline function. All computational work was carried out using MATLAB software. The numerical results show that only a few numbers of approximations can be used for numerical purpose with high degree of accuracy. It is observed that the absolute errors are better than the methods in [7, 9, 11, and 20]. It is also observed that our proposed method is well suited for the solution of higher order boundary value problems and reduces the computational work. Spline approximation method converges to exact solutions more rapidly as compared to the other method. Therefore, the present method is an accurate and reliable analytical technique for boundary value problems.

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