Tangles and Connectivity In Graphs

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Abstract - This project is a short introduction to the theory of tangles, both in graphs and general connectivity systems. An emphasis is put on the correspondence between tangles of order k and k-connected components. In particular, we prove that there is a one-to-one correspondence between the triconnected components of a graph and its tangles of order 3.

I. INTRODUCTION

Tangles, introduced by Robertson and Seymour in their graph monors series, have come to play an important role in structural graph theory.

Tangles describe highly connected regions in a graph. In a precise mathematical sense, they are "*dual*" to decompositions. Intuitively, a graph has a highly connected region described by a tangle if and only if it does not admit a decomposition along separators of low order.By decomposition I always mean a decomposition in a treelike fashion; formally, this is captured by the notions of tree decomposition or branch decomposition. However, tangles describe regions of a graph in an indirect and elusive way. This is why we use the unusual term "region" instead of "*subgraph*" or "*component*". The idea is that a tangle describes a region by pointing to it. A bit more formally, a tangle of order k assigns a "big side" to every separation of order less than k. The big side is where the (imaginary) region described by the tangle is of "big sides" to the separations is subject to certain consistency and nontriviality conditions, the "tangle axioms".

To understand why this way of describing a "region" is a good idea, let us review decompositions of graphs into their k-connected components. It is well known that every graph can be decomposed into its connected components and into its biconnected components. The former are the (inclusionwise) maximal connected subgraphs, and the latter the maximal 2-connected subgraphs. It is also well-known that a graph can be decomposed into its triconnectedcomponents, but the situation is more complicated here. Different from the triconnected components are not maximal 3-connected subgraphs; in fact they are not even subgraphs, but just topological subgraphs.

In general a graph does not have a reasonable decomposition into 4-connected components(neither into k-connected components for any $k \geq 5$), at least if these components are supposed to be 4-connected and some kind of subgraph. Consider the hexagonal grid.

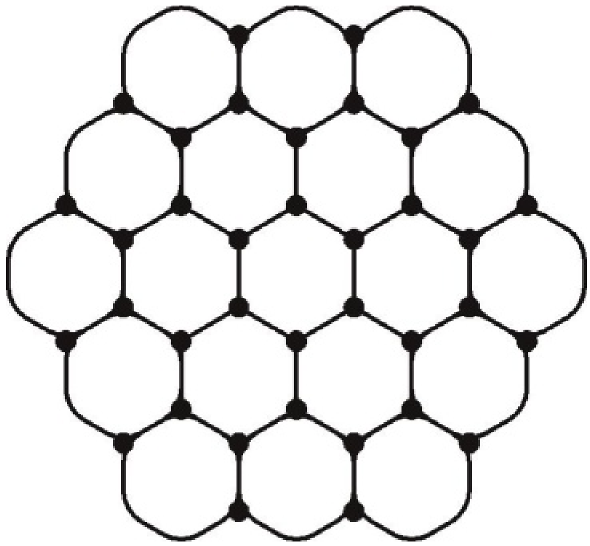

Fig : 1

It is 3-connected, but not 4-connected. In fact, for any two nonadjacent vertices there is a separator of order 3 separating these two vertices. Thus it may be possible for 4-connected components of a grid. But may be we need to adjust our view on connectivity: a hexagonal grid is fairly highly connected in a "global sense". All its low-order separations are very unbalanced. In particular, all separations of order 3 have just a single vertex on one side and all other vertices on the other side. This type of tangles are related to global connectivity. For example, there is a unique tangle of order 4 in the hexagonal grid: the big side of a separation of order 3 is obviously the side that contains all but one vertex. The "region" this tangle describes is just the grid itself. This does not sound particularly interesting, but the grid could be a subgraph of a larger graph, and then the tangle would identify it as a highly connected region within that graph. A key theorem about tangles is that every graph admits a canonical tree decomposition into its tangles of order k. This can be seen as a generalization of the decomposition of a graph into its 3-connected components. A different, but related generalization has been given.

The theory of tangles and decompositions generalizes from graphs to an abstract setting of connectivity systems. This includes nonstandard notions of connectivity on graphs, such as the "cut-rank" function, which leads to the notion of "rank width", and connectivity functions on other structures, for example matroids. Tangles give us an abstract notion of "k-connected components" for these connectivity systems. The canonical decomposition theorem to this abstract setting can be generalized from graphs.

This project is a short introduction to the basic theory of tangles, both for graphs and for general connectivity systems. We put a particular emphasis on the correspondence between tangles of order k and k-connected components of a graph for $k \leq 3$, which gives some evidence to the claim that for all k, tangles of order k may be viewed as a formalization of the intuitive notion of "k-connected component".

II. TANGLES IN A GRAPH AND COMPONENTS

In this project, we introduce tangles of graphs, giving a few examples, and review a few basic facts about tangles, all well-known and at least implicitly from fundamental work on tangles.

Let G be a graph. A G-tangle of order k is a family *T*of separations of Gsatisfying the following conditions.

(GT.0) The order of all separations (A,B) ∈T is less than k

(GT.1)For all separations (A,B) of G of order less than k, either

(A,B) ∈ Tor (B,A)∈ T

(GT.2)If (□₁,□₁), (□₂,□₂), (□₃,□₃) ∈ T then $A_1 \cup A_2 \cup A_3 \neq G$.

(GT.3)V(A)≠ V(G) for all (A,B)∈T

Observe that (GT.1) and (GT.2) imply that for all separations (A,B) of G of order less than k, exactly one of the separations (A,B), (B,A) is in T. We denote the order of a tangle *T* by ord (*T*).

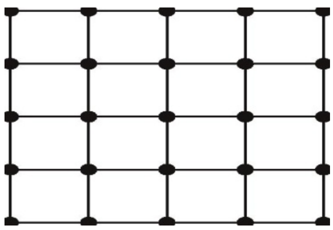

Fig : 2.1

EXAMPLE 2.1:

Let G be a graph and C⊆G a cycle. Let T be the set of all separations (A,B) of G of order 1 such that C⊆B. Then T is a G-tangle of order 2.

T trivially satisfies(GT.0). It satisfies (GT.1) , because for every separation (A,B) of G of order 1, either C⊆A or C⊆B. To see that T satisfies (GT.3).Let (□,□) ∈ *T*for i=1,2,3. Note that it may happen that

$V(A_1) \cup V(A_2) \cup V(A_3) = V(G)$ (if |C| = 3) .

However, no egde of C can be in E(□) for any i, because *C*⊆ □ and |□ ∩ □| ≤ 1.Hence $E(A_1) \cup E(A_2) \cup E(A_3) = E(G)$,which implies (GT.2).

Finally, T satisfies (GT.3), becauseV(*C*)*V* (*A*) = ∅for all (*A*,*B*) ∈ *T*.

EXAMPLE 2.2:

Let G be a graph and X⊆V(G) a clique in G. Note that for all separations (A,B) of G, either *X* ⊆*V* (*A*) or *X* ⊆*V* (*B*).For every k≥1, let *Tk* be the set of all separations (A,B) of G of order less than k such that *X* ⊆*V* (*B*).

If $k < \frac{2}{3}|X|+1$, the set *Tk* is a *G*-tangle of order *k*. We omit the proof, which is similar to the proof in the previous example. Instead, we prove that *Tk* is not necessarily a *G*-tangle if $k = 2|X| + 1$. To see this, let *G* be a complete graph of order 3*n*, $k := 2n + 1$, and $X := V(G)$.Suppose for contradiction that *Tk* is a *G*-tangle of order *k*. Partition *X* into three sets X_1, X_2, X_3 of size *n*. For $i = j$, Let □:= $G[X_i \cup X_j]$ and □:= *G*. Then(□, □) is a separation of *G* of order $2n < k$. By (GT.1) and (GT.3), we have(□, □)) ∈ *Tk*. However, $A_{12} \cup A_{13} \cup A_{23} = G$, and this contradicts (GT.2).

LEMMA2.1:

Let T be a G-tangle of order k

(1) If (A, B) is a separation of G with $|V(A)| < k$ then (A,B) ∈ *T*.

(2) If (A,B)∈ *T* and (A', B') is a separation of G of order < k such that $B' \supseteq B$, *then* (A', B') ∈

(3)If (A,B), (□′, □′) ∈T and ord (*A* ∪□′,B∩□′) < k then(*A* ∪□′,B∩□′) ∈T.

PROOF:

We leave the proofs of (1) and (2).To prove (3), let(*A*,*B*), (□′, □′) ∈T and ord(*A* ∪□′,*B* ∩□′) < *k*. By (GT.1),either (*A* ∪□′,*B* ∩□′) ∈T or (*B* ∪□′,A∩□′) ∈ *T*. As (*A* ∪□′) ∪ (*B* ∪□′) = *G*, by (GT.2) .we cannot have (*B* ∪□′,*A* ∩ □′) ∈ *T* .

LEMMA 2.2:

Let T be a G-tangle of order k. Then for every set S ⊆*V* (*G*) *of cardinality* |*S*| < *k there is a unique connected component C*(*T* , *S*) *of G\S suchthat for all separations* (*A*,*B*) *of G with V* (*A*) ∩ *V* (*B*) ⊆*S we have* (*A*,*B*) ∈*T* ⇐⇒*C*(*T* , *S*) ⊆*B*.

PROOF

Let *C*1, . . . , *Cm* be the set of all connected components of *G* \ *S*.

Forevery $I \subseteq [m]$, let □ : $\cup_{□\in□}$ □. We define a separation (□,□) of G as follows. □ is the graph with vertex set $S \cup V$ (□) and all edges that

have at least one end vertex in V (□), and □ is the graph with vertex set $S \cup V$(□$_{[□]}\backslash I$) and edge set $E(G)\backslash E$(□)). Note that V (□)$\cap V$ (□)) = S and

Thus ord(□,□) < k. Thus for all I, either (□,□) $\in$ Tor (□, □) $\in T$.

It follows From (GT.2) that (□, □) $\in$ T implies (□$_{[□]}\backslash I$, □$_{[□]}\backslash I$) $\in T$,

because ($G[S]$,G) $\in$ T and □ $\cup$□$_{[□]}\backslash I \cup G[S]$ = G. Furthermore, it follows from that (□,□), (□,□) $\in T$ implies (□$\cap J$, □$\cap J$) $\in$ T. By (GT.3)we have (□$_{[□]}$, □$_{[□]}$) $\in T$ and (□$_{\varnothing}$, □$_{\varnothing}$) $\in T$. Let $I \subseteq [m]$ be of minimum cardinality such that (□,□) $\in$ T. Since (□,□),(□,□) $\in$ Timplies (□$\cap J$, □$\cap J$) $\in T$, the minimum set I is unique. If $|I|$ = 1,then we let $C(T , S)$:= □ for the unique element $i \in I$. Suppose for contradiction that $|I|$ >1, and let $i \in I$. By the minimality of $|I|$ we have (□$_{\{□\}}$, □$_{\{□\}}$) $\in$ $T$and thus (□$_{[□]}\backslash\{i\}$,□$_{[□]}\backslash\{i\}$) $\in$ T. This implies (□$\backslash\{i\}$,□$\backslash\{i\}$) $\in T$, contradicting theminimality of $|I|$.

Let G be a graph. We say that subgraphs$C1, \ldots , Cm \subseteq G$ *touch* if there is a vertex $v \in \cap_{□=1}^{□}$ □(□) or an edge $e \in E(G)$ such that each □ contains atleast one end vertex of e. A family C of subgraphs of G *touches pairwise* if all □$_1$, □$_2$ $\in C$touch, and it *touches triplewise* if all □$_1$, □$_2$, □$_3$ $\in C$touch. A *vertexcover* (or *hitting set*) for C is a set $S \subseteq V(G)$ such that $S \cap V(C) = \varnothing$

THEOREM2.1

A graph G has a G-tangle of order k if and only ifthere is a family C of connected subgraphs of G that touches triplewise and has

no vertex cover of cardinality less than k.

PROOF:

In fact, defines a tangle of a graph G to be a family C of connected

subgraphs of G that touches triplewise and its order to be the cardinality of a

minimum vertex cover.

A *bramble* is a family C of connected subgraphs of G that touches pairwise. In this sense,a tangle is a special bramble.For the forward direction, let T be a G-tangle of order k.We let

$C := \{C(T , S) \mid S \subseteq V(G)$ with $|S| < k\}$.

C has no vertex cover of cardinality less than k, because if $S \subseteq V(G)$ with$|S| < k$ then $S \cap V(C(T , S)) = \varnothing$. It remains to prove that C touches triplewise.For i = 1, 2, 3, let $Ci \in C$ and $Si \subseteq V(G)$ with $|Si| < k$ such that □= $C(T$, □).

Let Bi be the graph with vertex set V (□) $\cup S$ and all edges of G that have atleast one vertex in V (□), and let Ai be the graph with vertex set $V(G) \backslash V$ (□)and the remaining edges of G. Since $C(T$, □) = □$\subseteq$□, we have (□,□) $\in T$.Hence □$_1 \cup$□$_2 \cup$□$_3$ = G by (GT.2), and this implies that □$_1$, □$_2$, □$_3$ touch.

For the backward direction, let C be a family of connected subgraphs of G that touches triplewise and has no vertex cover of cardinality less than k. Let T be the set of all separations (A,B) of G of order less than k such that $C \subseteq B \backslash V(A)$ for some $C \in C$ It is easy to verify that T is a G-tangle of

Order k.

Let T ,T be κ-tangles. If $T \subseteq T$, we say that T is an *extension* of T . The *truncation* of T to order $k \leq$ ord(T) is the set $\{(A,B) \in T / \text{ord}(A,B) < k\}$,

which is obviously a tangle of order k. Observe that if T is an extension of T then ord(T) $\leq$ ord(T), and T is the truncation of T to order ord(T).

III. CONCLUSION

Thus, we proved that the correspondence between tangles of order k and k-connected components. In particular, we proved that there is a one-to-one correspondence between the triconnected components of a graph and its tangles of order 3.

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