Dyadic Analysis of Curvature and Weyl Tensors

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Abstract-In the present paper some results are obtained Dyadic components of the curvature tensor also Dyadic analysis is done for Weyl Tensors. The obtained results are generalized form of well known results in the field of general theory of relativity.

Keywords: Dyadic Analysis, Weyl Tensor.

I. INTRODUCTION

In the present work an attempt has been made to study certain problems of general theory of relativity in terms of Dyadics. These dyadics are expressed with respect to the orthogonal space triad of an orthonormal space timetetrad set up at a point. Under certain orthogonal transformation of the tetrad field for which the time-like congruence do not change the three space-time congruences change so that the 3-dyadics and 3-vectors are introduced retain their character and interpretation. The equations set up in this formation bear a close relationship to corresponding equations in Newman- Penrose Spinor formalism with respect to a nulltetrad. According to following theorem by Geroch namely "A necessary and sufficient condition for a non-compact space time M to admit of a spin structure L that M has a global field of orthonormal tetrads", it is always possible to introduce orthonormal tetrads and hence 3- dyadics and 3-vectors as suggested above, whenever null-tetrads exist. We begin by introducing the notation of a dyadic and usual operation on it.

1.1 Differentiation in 3-space of auxiliary triad.

The intrinsic derivatives along the congruence of the orthonormal tetrad are given by

$$\left(\begin{smallmatrix}\alpha\ldots\ldots\\ T\ldots\ldots\mu;\sigma\end{smallmatrix}\right)\begin{smallmatrix}t\\ l\\ X\end{smallmatrix}\ldots\ldots\ldots\begin{smallmatrix}\mu\\ l\\ Y\end{smallmatrix}\;\begin{smallmatrix}\sigma\\ l\\ s\end{smallmatrix} = \begin{smallmatrix}t\ldots\ldots\\ T\ldots\ldots\ldots\mu;s\end{smallmatrix} + \begin{smallmatrix}..t\;q\;\ldots.\\ \ulcorner sq^{T\ldots u}\end{smallmatrix} + \ldots \begin{smallmatrix}\ldots..q\;t\;\ldots.\\ \ulcorner su^{T\ldots\ldots q}\end{smallmatrix}\ldots\ldots\ldots\ldots\ldots 1.1$$

Which are world scalars in which $\ulcorner rq^t$ play the same role as the coefficients of affine connections in Riemannian space. But while the latter are symmetric in the first two indices (for holonomic co-ordinates), $\ulcorner rs^t$ are antisymetric in the last two indices and have only 24 components compared to 40 in the case of former.

Now $T^{t\ldots\ldots\ldots..}_{\ldots\ldots u\ldots.}$ being a world scalar $T^{t\ldots\ldots}_{\ldots.u\ldots\ldots,[\mu\nu]=c}$ Hence successive intrinsic differentiation gives,

$$T^{t\ldots\ldots}_{\ldots..u\ldots\ldots,[rs]} = \ulcorner^{\ldots..\Sigma}_{[sr]}\, T^{t\ldots\ldots}_{\ldots.u\ldots.,q} = \Lambda^{q}_{.rs}\; T^{t\ldots\ldots}_{\ldots\ldots u\ldots..,q} \ldots\ldots\ldots 1.2$$

Intrinsic differentiation of the relation

$$\Lambda^{r}_{.st} = \frac{1}{2}\begin{pmatrix} r & & r\\ l_{\mu;\nu} & - & l_{\nu;\mu}\end{pmatrix} t^{l^\mu}\; s^{l^\nu}$$

and complete antisymmetrication with respect to Lorentz indices give

$$\Lambda^{r}_{.[str]} = 2\Lambda^{q}_{.[rs\Lambda^{r}_{.t}]q}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1.3$$

a set of 16 differential identities which are integrability conditions of $\Lambda^{r}_{.st}$ as the world - lines of a whole set of observers. It has been shown that vacuum space-times with petrov types I II or D weyl tensors can not be complex recurrent.

In chapters IV an V we consider the motion of currents and break up the strangled components of tensor

$$J_{\lambda\mu\nu} = R_{\nu[\lambda;\mu]} - \frac{1}{6}\, g_{\nu[\lambda^R,\mu]}$$

in two 3-vectors $\underline{J}$ and $\underline{K}$ and two traceless 3-dyauics J and K for gravitational fields interacting with matter or with electromagnetic fields. In the former case, we have considered the perfect null and in the latter, both null and non-null electromagnetic fields.

It is convenient to introduce a 3-vector operator to denote covariant differentiation in the 3-space of the auxiliary space-like congruences from equation 2.1:

$$\nabla_k \delta_{ij} = D_k \delta_{ij} + \delta^{.h}_{k.i} C_{hj} + \delta^{.h}_{k.j} C_{ih} \ldots\ldots\ldots\ldots\ldots 1.4$$

Where D_k are the ordinary derivatives in the 3-space and are components of the 3-vector operator $\underline{D}$ which define gradient, divergence and cunl. For the operator ∇, we have the following formulae. If V is a 3-vector and M be a symmetric 3-dyadics,

$$\underline{\nabla}\,\underline{v} = \underline{D}\,\underline{v} - \left[N - \frac{1}{2}(t_r N)i - \underline{L}\times 1\right]\times \underline{v}\ldots\ldots\ldots..1.5$$

$$\underline{\nabla}.\underline{v} = \underline{D}\,\underline{v} - 2\,\underline{L}\,.\underline{v}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1.6$$

$$\underline{\nabla}\times\underline{v} = \underline{D}\times\underline{v} - N.\underline{v} - \underline{L}\times\underline{v}\ldots\ldots\ldots\ldots\ldots 1.7$$

$$\underline{\nabla}\times M = \underline{D}\times M - MN - 2N.M - \underline{L}\times M + \underline{L}\,M\times 1 + \tfrac{1}{2}(t_r N)M + (t_r M)N + (N.M)1 - \tfrac{1}{2}(t_r N)(t_r M)1\ldots\ldots\ldots\ldots\ldots 1.8$$

$\underline{\nabla} \times M - M \times \underline{\nabla} = \underline{D} \times M - M \times \underline{D} - 3M.N - 3N.M - \frac{1}{2} \times M + M \times \underline{L} + (t_r N)M + 2(t_r M)N + 2(N.M)1 - (t_r N)(t_r M)1$ … … … … … … … … … … … … … … … … … … 1.9

$$\underline{\nabla}.M = \underline{D}.M - 3\underline{L}.M - N \times M + (t_r M) \text{ … …. } 1.10$$

2.2. Dyadic components of the curvature tensor:

The strangled components of the curvature tensor $R_{\lambda\mu\nu\rho}$ can be split into following manner.

R_{oioj} Containing two indices equal to zero.

R_{oijk} Containing one index equal to zero.

R_{hijk} Containing no index equal to zero.

Here indices are Lorentz indices denoting components with respect to

O N T. And h, i, j, k take the values 1, 2, 3 strangled components can but be calculated using

$$\Gamma^{[t|sr|,k]} = \frac{1}{2}\Gamma^{prq}\Gamma^{ts}_{..q} - \frac{1}{2}\Gamma^{trq}\Gamma^{ps}_{..q} + \Gamma^{[pt]q}\Gamma^{..sr}_{q} + \frac{1}{2}R^{strp} \text{ … … … … … … … … ….} 1.11$$

Where $[t|sr|p]$ means s and r are excluded from antisymmetrisation.

Here follow the component of curvature Tensor.

$$R^{0101} = \Gamma^{001,1} - \Gamma^{101,0} - \Gamma^{112}\Gamma^{00}_{-2} - \Gamma^{113}\Gamma^{00}_{-3} + \Gamma^{012}\Gamma^{10}_{-2} + \Gamma^{013}\Gamma^{10}_{..3} - \Gamma^{101}\Gamma^{.01}_{1} - \Gamma^{102}\Gamma^{.01}_{2} - \Gamma^{103}\Gamma^{.01}_{3} + \Gamma^{012}\Gamma^{.01}_{2} + \Gamma^{013}\Gamma^{.01}_{3}$$

$R_{0101} = -s_{11} + a_{1,1} + a_{2(N_{13}-l_2)} - a_{3(N_{12}+l_3)} - s_{11}^2 - s_{12}s_{12} - s_{13}s_{13} + 2(\omega^3 s_{12} - \omega^2 s_{13}) + a_1^2 + (\Lambda^2)^2 + (\Lambda^3)^2 = Q_{11}$

$R_{0203} = R_{0302} = -s_{23} + \frac{1}{2}\{a_{2,3} + a_{3,2} + a_1(N_{22} - N_{33}) - a_2(N_{12} - l_3) + a_3(N_{12} + l_2)\} - s_{12}s_{13} - s_{23}(s_{22} + s_{33}) + s_{12}\omega^2 - s_{22}\omega^1 - \omega^3 s_{13} + \omega^1 s_{33} + a_2 a_3 - \Lambda^2\Lambda^3 = Q_{23}$

$R_{0123} = \{s_{12,3} - s_{13,2}\} + (s_{11}N_{11} + s_{12}N_{12} + s_{13}N_{13}) + l_2 S_{13} - l_3 s_{12} - \frac{1}{2}(N_{11} + N_{22} + N_{33})s_{11} - (S_{11} + S_{22} +$ *S33N11−S11N11+2S12N12+2S13N13+S22N22+2S23N23+S33N33+12S11+S22+S33N11+N22+N33+Λ1,−N12+l3Λ3+(N13−l2)Λ2+2a1Λ1−a1Λ1+a2Λ2+a3Λ3=B11*

$R_{0112} = S_{11,2} - S_{12,1} + \Lambda^3_{11} + (N_{23} + L_1)(S_{12} + \Lambda^3) + \frac{1}{2}(N_{11} - N_{22} + N_{33})(S_{13} - \Lambda^2) - (N_{13} - L_2)S_{22} + (N_{12} + L_3)(S_{23} + \Lambda^1) - a_1(S_{12} - \Lambda^3) + (N_{23} + L_1)(S_{12} - \Lambda^3) + \frac{1}{2}(N_{11} - N_{22} + N_{33})(S_{13} + \Lambda^2) + a_1(S_{12} + \Lambda^3) + (N_{13} - L_2)S_{11} + \frac{1}{2}(N_{11} + N_{22} + N_{33})(S_{13} + \Lambda^2) = B_{13} + t^2$

$R_{0221} = \delta_{22,1} - \delta_{12,2} - \Lambda^3{}_{12} - (N_{13} - l_2)(S_{12} - \Lambda^3) + \frac{1}{2}(N_{11} - N_{22} + N_{33})(S_{23} + \Lambda^1) + (N_{23} + l_1)S_{11} - (N_{12} - l_3)(S_{13} - \Lambda^2) - a_2(S_{12} + \Lambda^3) - (N_{13} - l_2)(S_{12} + \Lambda^3) + \frac{1}{2}(N_{11} - N_{22} + N_{33})(S_{23} - \Lambda^1) = -B_{23} + t^1$

$R_{0331} = \delta_{33,1} - \delta_{13,3} + \Lambda^2{}_{13} + (N_{12} + l_3)(S_{13} + \Lambda^2) - \frac{1}{2}(N_{11} - N_{22} - N_{33})(S_{23} - \Lambda^1) - (N_{23} - l_1)S_{11} + (N_{13} + l_2)(S_{12} + \Lambda^3) - a_3(S_{13} - \Lambda^2) + (N_{12} + l_3)(S_{13} - \Lambda^2) - \frac{1}{2}(N_{11} - N_{22} - N_{33})(S_{23} + \Lambda^1) + a_3(S_{13} + \Lambda^2) + (N_{23} - l_1)\delta_{33} + \frac{1}{2}(N_{11} + N_{22} - N_{33})(S_{23} + \Lambda^1) = B_{23} + t^1$

$R_{2323} = \Gamma 223{,}3 - \Gamma 323{,}2 + \Gamma 330\,\Gamma 220 - \Gamma 331\,\Gamma 221 - \Gamma 23\,\Gamma 320 + \Gamma 231\,\Gamma 321 + \Gamma 320\,\Gamma 023 - \Gamma 321\,\Gamma 123 - \Gamma 32\,\Gamma 323 - \Gamma 23\,\Gamma 023 + \Gamma 23\,\Gamma 123 + \Gamma 232\,\Gamma 223$

$= -(N_{12} - l_3)_{13} + (N_{13} + l_2)_{12} + S_{22}S_{33} + (N_{23} - l_1)(N_{23} + l_1) - (S_{23} - \Lambda^1)(S_{23} + \Lambda^1) - \frac{1}{4}(N_{11} - N_{22} +$ *N33N11+N22−N33−ω1S23−Λ1−14N11+N22−N33N11−N22−N33−N13+l22+ω1S23+Λ1−14N11−N22+N33N11−N22−N33−N12−l32*

$= \frac{1}{2}\{\nabla \times N - N \times \nabla - (\nabla L + L\nabla)\}_{,1} + \nabla L + L_1 L_1 + \frac{1}{2}(S \times S)_{11} + \Lambda^1\Lambda^1 + \omega^1\Lambda^1 + \Lambda^1\omega^1 + \frac{1}{4}\left(N_{11}{}^2 - N_{22}{}^2 - N_{33}{}^2 +\right.$ *2N22N33−N232*

$= \frac{1}{2}\{\nabla \times N - N \times \nabla - (\nabla L + L\nabla)\}_{11} + L_1 L_1 + \frac{1}{2}(S^{\times}_{\times}S)_{11} + \Lambda^1\Lambda^1 + 2\omega^1\Lambda^1 + \frac{1}{2}(t_r N)N_{11} + \frac{1}{2}(N^{\times}_{\times}N)_{11} + \left\{\nabla.L - \frac{1}{4}(t_r N)^2\right\} = -P_{11}$

$$R_{3112} = -(N_{12}+L_3)_{,2} - \frac{1}{2}(N_{11}-N_{22}+N_{33})_{,1} + (S_{12}-\Lambda^3)(S_{13}-\Lambda^2) + \frac{1}{2}(N_{23}+L_1)(N_{11}-N_{22}-N_{33})$$
$$- S_{11}(S_{12}+\Lambda^1) - (N_{13}-L_2)(N_{12}-L_3) - \omega^2(S_{12}-\Lambda^3) + \frac{1}{2}(N_{23}+L_1)(N_{11}-N_{22}+N_{33})$$
$$-\frac{1}{2}(N_{11}-N_{22}+N_{33})(N_{23}-L_1) + \omega^2(S_{12}+\Lambda^3) - (N_{13}-L_2)(N_{12}+L_3)$$
$$+\frac{1}{2}(N_{23}-L_1)(N_{11}-N_{22}-N_{33}) = -P_{23}$$

$R_{2113} = (N_{13}-L_2)_{,3} + \frac{1}{2}(N_{11}+N_{22}-N_{33})_{,1} + (S_{13}+\Lambda^2).(S_{12}+\Lambda^3) + \frac{1}{2}(N_{23}-L_1)(N_{11}-N_{22}-N_{33}) -$ $S_{11}(S_{23}-\Lambda^1) - (N_{12}+L_3)(N_{13}+L_2) + \omega^3(S_{13}+\Lambda^2) + \frac{1}{2}(N_{23}-L_1)(N_{11}+N_{22}-N_{33}) - \frac{1}{2}(N_{23}+L_1)(N_{11}+N_{22}-$
N33−ω3S13−Λ2−N12+L3N13−L2+12N23+L1(N11−N22−N33)

$R_{3112} = \frac{1}{2}(R_{3112}+R_{2113}) = \frac{1}{2}\{\nabla\times N - N\times\nabla - (\nabla L + L\nabla)\} + L_2L_3 + \frac{1}{2}(S\overset{\times}{\times}S)_{23} + \Lambda^2\Lambda^3 + \omega^2\omega^3 + \Lambda^2\omega^2 +$ $\frac{1}{2}(N_{11}+N_{22}+N_{33})N_{23} + (N_{12}N_{13}-N_{11}N_{23}) = -P_{23}$

$$R_{ojok} = Q_{jk},$$

$Q = -s + \frac{1}{2}\left(\underline{\nabla}\,\underline{a} + \underline{a}\,\underline{\nabla}\right) - s.s + \underline{a}\,\underline{a} + s\times\underline{\omega} - \underline{\omega}\times s - \underline{\Lambda}\,\underline{\Lambda} + (\underline{\Lambda}.\underline{\Lambda})\,1$

Is a symmetric dyadic.

$$R_{0123} = B_{11}, R_{0131} = B_{12} - t^3, R_{0112} = B_{13} + t^2$$

$R_{0233} = B_{12} + t^3, R_{0231} = B_{22}, R_{0212} = B_{23} - t^1\; R_{0323} = B_{13} + t^1,\; R_{0331} = B_{23} + t^2, R_{0312} = B_{33}$

Where

$$B = -\frac{1}{2}\left(\underline{\nabla}\times S - S\times\underline{\nabla}\right) + \frac{1}{2}\left(\underline{\nabla}\,\underline{\Lambda} + \underline{\Lambda}\,\underline{\nabla}\right) + \underline{a}\,\underline{\Lambda} + \underline{\Lambda a} - \left(\underline{a}.\underline{\Lambda}\right)1$$

Is a traceless symmetric dyadic and

$$\underline{t} = -\frac{1}{2}\underline{\nabla}\,S + \frac{1}{2}\underline{\nabla}(t_rS) - \underline{\nabla}\times\underline{\Lambda} + 2(\underline{\Lambda}\times\underline{a})$$
$$R_{2323} = -P_{11}, R_{2331} = -P_{12}, R_{2312} = -P_{13}$$
$$R_{3131} = -P_{22}, R_{3112} = -P_{23}, R_{1212} = -P_{33}$$

Where

$P = -\frac{1}{2}\left(\underline{\nabla}\times N - N\times\underline{\nabla}\right) + \frac{1}{2}\left(\underline{\nabla}\,\underline{L} + \underline{L}\,\underline{\nabla}\right) - \frac{1}{2}(t_rN)N - \frac{1}{2}N\overset{\times}{\times}N - \frac{1}{2}S\overset{\times}{\times}S - \underline{\Lambda}\,\underline{\Lambda} - \underline{\omega}\,\underline{\Lambda} - \underline{\Lambda}\,\underline{\omega} - \underline{L}\,\underline{L} - \left\{\underline{\nabla}\,\underline{\Lambda} - \frac{1}{4}(t_rN)^2\right\}1$

Is a symmetric dyadic. We can write

$$P = -\left(E + \frac{1}{2}S\overset{\times}{\times}S\right) + \underline{\Lambda}\,\underline{\Lambda} + \underline{\omega}\,\underline{\Lambda} + \underline{\Lambda}\,\underline{\omega}),$$

Where $E = \frac{1}{2}\left(\underline{\nabla}\times N - N\times\underline{\nabla}\right) - \frac{1}{2}\left(\underline{\nabla}\,\underline{L} + \underline{L}\,\underline{\nabla}\right) + \frac{1}{2}(t_rN)N + \frac{1}{2}N\overset{\times}{\times}N + \underline{L}\,\underline{L} + \left\{\underline{\nabla}\,\underline{L} - \frac{1}{4}(t_rN)^2\right\}1$

We have $t_rp = \frac{1}{2}N{:}N - \frac{1}{4}(t_rN)^2 - 2\underline{\nabla}.\underline{L} - \underline{L}\,\underline{L} - \underline{\Lambda}\,\underline{\Lambda} - 2\underline{\omega}\,\underline{\Lambda}$

In case $o^{e^{\mu}}$ determine a normal congruence ($\underline{\Lambda} = 0$)

$$E = -P - \frac{1}{2}S\overset{\times}{\times}S$$

and its six components give the six components of curvature tensor of the Riemannian 3-space orthogonal to $o^{e^{\mu}}$. We also have, in this case

$\underline{\nabla}\,.\underline{\mathrm{E}} = 0$

3. Representation of curvature tensor in bi-vector space :

Strangled components of the Ricci tensor are given by

$R_{rs} = R.^{t}{}_{rst}$

these can be spilt into R_{ij} , R_{0i} , R_{00} ; $\quad i,j = 1,2,3$

$R_{11} = R^0{}_{.110} + R^2{}_{.112} + R^3{}_{.113} = -R_{0110} + R_{2112} + R_{3113}$

$$= R_{0101} + R_{2112} + R_{3113} = Q_{11} + P_{22} + P_{33} = Q_{11} - P_{11} + t_r P$$

$$R_{12} = R^0{}_{.120} + R^3{}_{.123} = R_{0102} + R_{3123} = Q_{12} - P_{12}$$

$$R_{01} = R^2{}_{.012} + R^3{}_{.013} = R_{.2012} + R_{3013} = R_{1221} + R_{0331} = 2t_1$$

$$R_{00} = R^1{}_{.001} + R^2{}_{.002} + R^3{}_{.003} = R_{1001} + R_{2012} + R_{3003}$$

$$= -(R_{0101} + R_{0202} + R_{0303}) = -(Q_{11} + Q_{22} + Q_{33}) = -t_r Q$$

Hence the ten strangled components of the Ricci tensor correspond to a symmetric 3-dyadic $Q - P + (t_r P)1$

a 3-vector $\underline{t}$

and a scalar $-(t_r Q)$

The scalar curvature R can be expressed as

$$R = R^r{}_{.r} = \eta^{rs} R_{sr} = -R_{00} + R_{11} + R_{22} + R_{33} = 2(t_r P + t_r Q)$$

In the Orthonormal tetrad frame, Einstein's equations take the form

$$2T_{rs} = R_{rs} - \frac{1}{2} R \eta_{rs}$$

Hence,

$$2T_{11} = R_{11} - \frac{1}{2} R = Q_{11} - P_{11} - t_r Q \ , 2T_{12} = R_{12} = Q_{12} - P_{12}$$

$$2T_{01} = R_{01} = 2t_1$$

$$2T_{00} = R_{00} + \frac{1}{2} R = -t_r Q + t_r P + t_r Q = t_r P$$

Hence the energy - momentum tensor breaks up into symmetric 3-dyadic $T = \frac{1}{2}[Q - P - (t_r Q)1]$

3 - vector $\underline{t}$

and scalar $P = \frac{1}{2}(t_r P)$

in the 3-space of auxiliary congruences

we know that curvature tensor $R_{\lambda \mu r P}$ is represented in a 6-dimentional bivector space by introducing the collective indices

$$01 \longleftrightarrow 1, 02 \longleftrightarrow 2 \ , 03 \longleftrightarrow 3$$

$$23 \longleftrightarrow 4 \ , 31 \longleftrightarrow 5 \ , 12 \longleftrightarrow 6$$

In terms of Q, P, B and $\underline{t}$ the matrix (R_{AB}), $A, B = 1,2,3, - - -6$ has the representation

$$R_{AB} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & B_{11} & B_{12} - t^3 & B_{13} + t^2 \\ Q_{12} & Q_{22} & Q_{23} & B_{12} + t^3 & B_{22} & B_{23} - t^1 \\ Q_{13} & Q_{23} & Q_{33} & B_{13} - t^2 & B_{23} + t^1 & B_{33} \\ B_{11} & B_{12} + t^3 & B_{13} - t^2 & -P_{11} & -P_{12} & -P_{13} \\ B_{12} - t^3 & B_{22} & B_{23} + t^1 & -P_{12} & -P_{22} & -P_{23} \\ B_{13} + t^2 & B_{23} - t^1 & B_{33} & -P_{13} & -P_{23} & -P_{33} \end{pmatrix}$$

i.e. $R_{AB} = \begin{pmatrix} Q & \mu \\ \mu^T & -P \end{pmatrix}$ where $\mu = B + \underline{t} \times 1$

Introducing the traceless symmetric dyadic

$$A = \frac{1}{2}\left[P + Q - \frac{1}{3}(t_r P + t_r Q)1\right]$$

we have

$$Q = T + A - \left(P - \frac{1}{3} R\right) 1, -P = T - A - \left(P - \frac{1}{6} R\right) 1$$

where

$$R = 2(t_r T - P)$$

Hence (R_{AB}) may be represented by T , A and B as

$$R_{AB} = \begin{pmatrix} A + T - \left(P - \frac{1}{3}R\right)1 & B + \underline{t} \times 1 \\ B - \underline{t} \times 1 & A - T + \left(P - \frac{1}{6}R\right)1 \end{pmatrix}$$

2.4. Dyadic analysis of Weyl tensor:

In terms of the Ortho-normal tetrad corresponding to the decomposition

$$R_{\lambda\mu rP} = C_{\lambda\mu rP} + g_{\lambda[r}R_{P]\mu} + R_{\lambda[r}g_{P]\mu} - \frac{1}{3}Rg_{\lambda[r}g_{P]\mu}$$

We have

$$R_{rstu} = C_{rstu} + \left(\eta_{r[u}E_{t]s} - \eta_{s[u}E_{t]r}\right) + \frac{1}{12}R\left(\eta_{r[u}\eta_{t]s} - \eta_{s[u}\eta_{t]r}\right)$$

Where C_{rstu} are the strangled components of Weyl tensor and

$$E_{rs} = R_{rs} - \frac{1}{4}R\eta_{rs}$$

we have

$$R_{0101} = C_{0101} + \frac{1}{2}E_{11} - \frac{1}{2}E_{00} + \frac{1}{24}R$$

$$= C_{0101} + \frac{1}{2}R_{11} - \frac{1}{2}R_{00} - \frac{1}{6}R$$

$$= C_{0101} + \frac{1}{2}(Q_{11} - P_{11} + t_rP) + \frac{1}{2}t_rQ - \frac{1}{3}(t_rP + t_rQ)$$

$$C_{0101} = Q_{11} - \frac{1}{2}(Q_{11} - P_{11}) + \frac{1}{3}(t_rP + t_rQ) - \frac{1}{2}(t_rP + t_rQ) = \frac{1}{2}(P_{11} + Q_{11}) + \frac{1}{6}(t_rP + t_rQ)$$

$$= \frac{1}{2}(P_{11} + Q_{11}) - \frac{1}{6}(t_rP + t_rQ)$$

$$R_{0203} = C_{0203} + \frac{1}{2}E_{23}$$

$$= C_{0203} + \frac{1}{2}R_{23}$$

$$Q_{23} = C_{0203} + \frac{1}{2}(Q_{23} - P_{23})$$

$$C_{0203} = \frac{1}{2}(P_{23} + Q_{23})$$

$$R_{0123} = C_{0123}$$

$$\therefore\ C_{0123} = B_{11}$$

$$R_{0221} = C_{0221} + \frac{1}{2}E_{10} = C_{0221} + \frac{1}{2}R_{01}$$

$$\therefore\ C_{0221} = t_1 - B_{23} - t_1 = -B_{23}$$

$$R_{2323} = C_{2323} - \frac{1}{2}E_{33} - \frac{1}{2}E_{22} - \frac{1}{12}R$$

$$= C_{2323} - \frac{1}{2}R_{33} - \frac{1}{2}R_{22} + \frac{1}{4}R - \frac{1}{12}R$$

$$= C_{2323} - \frac{1}{2}R_{33} - \frac{1}{2}R_{22} + \frac{1}{6}R$$

$$C_{2323} = -P_{11} + \frac{1}{2}(Q_{33} - P_{33} + t_rP) + \frac{1}{2}(Q_{22} - P_{22} + t_rP) - \frac{1}{3}(t_rP + t_rQ)$$

$$= -P_{11} + \frac{1}{2}P_{11} - \frac{1}{2}t_rP - \frac{1}{2}Q_{11} + \frac{1}{2}t_rQ + t_rP - \frac{1}{3}(t_rP + t_rQ)$$

$$= -\frac{1}{2}(P_{11} + Q_{11}) + \frac{1}{3}(t_rP + t_rQ)$$

$$R_{3112} = C_{3112} + \frac{1}{2}E_{23} = C_{3112} + \frac{1}{2}R_{23}$$

$$= C_{3112} + \frac{1}{2}(Q_{23} - P_{23})$$

$$C_{3112} = -P_{23} - \frac{1}{2}(Q_{23} - P_{23}) = -\frac{1}{2}(P_{23} + Q_{23})$$

$\therefore\ C_{0101} = -C_{2323} = \frac{1}{2}(P_{11} + Q_{11}) - \frac{1}{6}(t_r P + t_r Q) = A_{11}$

$C_{0203} = -C_{3112} = A_{23}$

$A = \frac{1}{2}(P + Q) - \frac{1}{6}(t_r P + t_r Q)1$

Hence the symmetric dyadic A and B together with $t_r A = t_r B = 0$

give the ten independent strangled components of weyl tensor.

5. Petrov type classification for A and B :

The five dyad components $\underline{\overline{\psi}}_0, \underline{\overline{\psi}}_1, \underline{\overline{\psi}}_2, \underline{\overline{\psi}}_3,$ *and* $\underline{\overline{\psi}}_4$ of weyl spinor

corresponding to $C_{\lambda\mu rP}$ are

$$\underline{\overline{\psi}}_0 = -C_{\alpha\beta\gamma\delta}\ l^\alpha m^\beta l^\gamma m^\delta$$

$$\underline{\overline{\psi}}_1 = -C_{\alpha\beta\gamma\delta}\ l^\alpha n^\beta l^\gamma m^\delta$$

$$\underline{\overline{\psi}}_2 = -\frac{1}{2} C_{\alpha\beta\gamma\delta}\ (l^\alpha n^\beta l^\gamma n^\delta + l^\alpha n^\beta m^\gamma \overline{m}^\delta)$$

$$\underline{\overline{\psi}}_3 = C_{\alpha\beta\gamma\delta}\ l^\alpha n^\beta n^\gamma \overline{m}^\delta$$

$$\underline{\overline{\psi}}_4 = -C_{\alpha\beta\gamma\delta}\ n^\alpha \overline{m}^\beta n^\gamma \overline{m}^\delta$$

From these components, we have

$$-\underline{\overline{\psi}}_0 = C_{\alpha\beta\gamma\delta}\ l^\alpha m^\beta l^\gamma m^\delta \qquad (L_1)$$

i.e.=$-4\underline{\overline{\psi}}_0 = C_{\alpha\beta\gamma\delta} \left(\begin{matrix} e^\alpha \\ 0 \end{matrix} + \begin{matrix} e^\alpha \\ 1 \end{matrix}\right)\left(\begin{matrix} e^\beta \\ 2 \end{matrix} - i\begin{matrix} e^\beta \\ 3 \end{matrix}\right)\left(\begin{matrix} e^\gamma \\ 0 \end{matrix} + \begin{matrix} e^\gamma \\ 1 \end{matrix}\right)$

$= C_{\alpha\beta\gamma\delta} \left\{ \begin{matrix} e^\alpha e^\beta e^\gamma e^\delta \\ 0\ 2\ 0\ 2 \end{matrix} + \begin{matrix} e^\alpha e^\beta e^\gamma e^\delta \\ 0\ 2\ 1\ 2 \end{matrix} + \begin{matrix} e^\alpha e^\beta e^\gamma e^\delta \\ 1\ 2\ 0\ 2 \end{matrix} + \begin{matrix} e^\alpha e^\beta e^\gamma e^\delta \\ 1\ 2\ 1\ 2 \end{matrix} - \left(\begin{matrix} e^\alpha e^\beta e^\gamma e^\delta \\ 0\ 3\ 0\ 3 \end{matrix} + \begin{matrix} e^\alpha e^\beta e^\gamma e^\delta \\ 0\ 3\ 1\ 3 \end{matrix} + \begin{matrix} e^\alpha e^\beta e^\gamma e^\delta \\ 1\ 3\ 0\ 3 \end{matrix} + \right.\right.$
e1αe3βe1γe3δ− ie0αe2βe0γe3δ+e0αe2βe1γe3δ+e1αe2βe0γe3δ+e1αe2βe1γe3δ+e0αe3βe0γe2δ+e0αe3βe1γe2δ+e1αe3βe0γe2δ+e1αe3βe1γe2δ

$=C_{0202} + 2C_{0212} + C_{1212} - C_{0203} - 2C_{0313}$

$-C_{1313} - i(2C_{0203} + 2C_{0213} + 2C_{1203} + 2C_{1213})$

$= C_{0202} + 2C_{0212} - 2C_{0313} - C_{0303} + C_{1212}$

$-C_{1313} - 2i(C_{0203} + C_{0213} + C_{1203} + C_{1213})$

$$2\underline{\overline{\psi}}_0 = (-A_{22} + A_{33} - 2B_{23}) + i(-B_{22} + B_{33} + 2A_{23})$$

Similarly,

$2\underline{\overline{\psi}}_1 = -(A_{12} + B_{13}) + i(A_{13} - B_{12}) \dots\dots\dots\dots (L_2)$

$2\underline{\overline{\psi}}_2 = -A_{11} - iB_{11} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (L_3)$

$2\underline{\overline{\psi}}_3 = -(A_{12} - B_{13}) - i(A_{13} + B_{12}) \dots\dots\dots\dots (L_4)$

$$2\underline{\overline{\psi}}_4 = (-A_{22} + A_{33} + 2B_{23}) + i(-B_{22} + B_{33} - 2A_{23}) .. (L_5)$$

Hence

$A_{11} = -\left(\underline{\overline{\psi}}_2 + \underline{\overline{\overline{\psi}}}_2\right), A_{12} = -\frac{1}{2}\left(\underline{\overline{\psi}}_1 + \underline{\overline{\overline{\psi}}}_1 + \underline{\overline{\psi}}_3 + \underline{\overline{\overline{\psi}}}_3\right)$

$A_{13} = -\frac{i}{2}\left(\underline{\overline{\psi}}_1 - \underline{\overline{\overline{\psi}}}_1 - \underline{\overline{\psi}}_3 + \underline{\overline{\overline{\psi}}}_3\right),$

$A_{22} = \frac{1}{2}\left(\underline{\overline{\psi}}_2 + \underline{\overline{\overline{\psi}}}_2\right) - \frac{1}{4}\left(\underline{\overline{\psi}}_0 + \underline{\overline{\overline{\psi}}}_0 - \underline{\overline{\psi}}_4 + \underline{\overline{\overline{\psi}}}_4\right),$

$A_{23} = -\frac{i}{4}\left(\underline{\overline{\psi}}_0 - \underline{\overline{\overline{\psi}}}_0 - \underline{\overline{\psi}}_4 + \underline{\overline{\overline{\psi}}}_4\right),$

$A_{33} = \frac{1}{2}(\underline{\overline{\psi}}_2 + \underline{\overline{\overline{\psi}}}_2) + \frac{1}{4}\left(\underline{\overline{\psi}}_0 + \underline{\overline{\overline{\psi}}}_0 + \underline{\overline{\psi}}_4 + \underline{\overline{\overline{\psi}}}_4\right),$

$B_{11} = i\left(\underline{\overline{\psi}}_2 - \underline{\overline{\overline{\psi}}}_2\right),$

$B_{12} = \frac{i}{2}\left(\underline{\overline{\psi}}_1 - \underline{\overline{\overline{\psi}}}_1 + \underline{\overline{\psi}}_3 - \underline{\overline{\overline{\psi}}}_3\right),$

$B_{13} = -\frac{1}{2}\left(\underline{\overline{\psi}}_1 + \underline{\overline{\overline{\psi}}}_1 - \underline{\overline{\psi}}_3 - \underline{\overline{\overline{\psi}}}_3\right),$

$B_{22} = -\frac{i}{2}\left(\underline{\overline{\psi}}_2 - \underline{\overline{\overline{\psi}}}_2\right) + \frac{i}{4}\left(\underline{\overline{\psi}}_0 - \underline{\overline{\overline{\psi}}}_0 + \underline{\overline{\psi}}_4 - \underline{\overline{\overline{\psi}}}_4\right),$

$B_{23} = -\frac{1}{4}\left(\underline{\overline{\psi}}_0 + \underline{\overline{\overline{\psi}}}_0 - \underline{\overline{\psi}}_4 - \underline{\overline{\overline{\psi}}}_4\right),$

$B_{33} = -\frac{i}{2}\left(\underline{\overline{\psi}}_2 - \underline{\overline{\overline{\psi}}}_2\right) - \frac{i}{4}(\underline{\overline{\psi}}_0 - \underline{\overline{\overline{\psi}}}_0 + \underline{\overline{\psi}}_4 - \underline{\overline{\overline{\psi}}}_4)$

We can choose the null tetrad so that $\underline{\overline{\psi}}_0 = 0$

(l^μ is a principal null vector of Weyl tensor). Then setting

$$\underline{\overline{\psi}}_2 + \underline{\overline{\overline{\psi}}}_2 = \tilde{a}\,, \qquad \underline{\overline{\psi}}_4 + \underline{\overline{\overline{\psi}}}_4 = 4a\,, \qquad i\left(\underline{\overline{\psi}}_1 - \underline{\overline{\overline{\psi}}}_1 + \underline{\overline{\psi}}_3 - \underline{\overline{\overline{\psi}}}_3\right)$$

$i\left\{\left(\underline{\overline{\psi}}_1 - \underline{\overline{\overline{\psi}}}_1\right) - \left(\underline{\overline{\psi}}_3 - \underline{\overline{\overline{\psi}}}_3\right)\right\} = -2\overline{b}\,,\ i\left(\underline{\overline{\psi}}_4 - \underline{\overline{\overline{\psi}}}_4\right) = 4c$

$i\left(\underline{\overline{\psi}}_2 - \underline{\overline{\overline{\psi}}}_2\right) = -2\tilde{c}\,, \underline{\overline{\psi}}_1 + \underline{\overline{\overline{\psi}}}_1 + \underline{\overline{\psi}}_3 + \underline{\overline{\overline{\psi}}}_3 = -2d$

$\underline{\overline{\psi}}_1 + \underline{\overline{\overline{\psi}}}_1 - \underline{\overline{\psi}}_3 - \underline{\overline{\overline{\psi}}}_3 = -2\tilde{d}$,

we have the following representation for A and B.

$A = -2\tilde{a}\,\underline{uu} + d\left(\underline{uv} + \underline{v}\,\underline{u}\right) + \tilde{b}\left(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u}\right) + (\tilde{a} - a)\underline{v}\,\underline{v} + c\left(\underline{v}\,\underline{\omega} + \underline{\omega}\,\underline{v}\right) + (\tilde{a} + a)\underline{\omega}\,\underline{\omega}$; $B = -2\tilde{c}\,\underline{u}\,\underline{u} + b\left(\underline{uv} + \underline{vu}\right) + \overline{d}\left(\underline{u}\,\underline{\omega} + \underline{\omega u}\right) + (\tilde{c} + c)\underline{v}\,\underline{v} + a\left(\underline{v}\,\underline{\omega} + \underline{\omega}\,\underline{v}\right) + (\tilde{c} - c)\underline{\omega}\,\underline{\omega}$

For different Petrov types of Weyl tensor, we have for A and B,

Type -1: $\underline{\overline{\psi}}_0 = \underline{\overline{\psi}}_4 = 0$(Two of the principal null vectors of $C_{\lambda\mu rp}$ are l^μ and n^μ)

$$A = -2\tilde{a}\,\underline{uu} + d\left(\underline{uv} + \underline{vu}\right) + \tilde{b}\left(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u}\right) + \tilde{a}\underline{v}\,\underline{v} + \tilde{a}\,\underline{\omega}\,\underline{\omega}$$

$$B = -2\underline{\tilde{c}}\,\underline{u}\,\underline{u} + b\left(\underline{u}\,\underline{v} + \underline{v}\,\underline{u}\right) + d\left(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u}\right) + \tilde{c}\underline{v}\,\underline{v} + \tilde{c}\,\underline{\omega}\,\underline{\omega}$$

Type-2: $\underline{\overline{\psi}}_0 = \underline{\overline{\psi}}_1 = \underline{\overline{\psi}}_4 = 0$($l^\mu$ A repeated principal null vector and n^μ a second principal null vector).

$$A = -2\tilde{a}\underline{u}\,\underline{u} + d\left(\underline{u}\,\underline{v} + \underline{v}\,\underline{u}\right) + b\left(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u}\right) + \tilde{a}\,\underline{v}\,\underline{v} + \tilde{a}\,\underline{\omega}\,\underline{\omega}$$

$$B = -2\tilde{c}\underline{uu} + b\left(\underline{u}\,\underline{v} + \underline{v}\,\underline{u}\right) - d\left(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u}\right) + \tilde{c}\underline{v}\,\underline{v} + \tilde{c}\underline{\omega}\,\underline{\omega}$$

Type-D: $\underline{\overline{\psi}}_0 = \underline{\overline{\psi}}_1 = \underline{\overline{\psi}}_3 = \underline{\overline{\psi}}_4 = 0$ (principal null direction of $C_{\lambda\mu rp}$ are$l^\mu, l^\mu, n^\mu, n^\mu$).

$A = -2\tilde{a}\underline{u}\,\underline{u} + \tilde{a}\underline{v}\,\underline{v} + \tilde{a}\,\underline{\omega}\,\underline{\omega}$; $B = -2\overline{c}\underline{uu} + \tilde{c}\underline{v}\,\underline{v} + \tilde{c}\underline{\omega}\,\underline{\omega}$

Type-3: $\underline{\overline{\psi}}_0 = \underline{\overline{\psi}}_1 = \underline{\overline{\psi}}_2 = \underline{\overline{\psi}}_4 = 0$ (principal null directions of $C_{\lambda\mu rp}$ are $l^\mu, l^\mu, l^\mu, n^\mu$).

$A = d\left(\underline{u}\,\underline{v} + \underline{v}\,\underline{u}\right) + b(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u})$; $B = b\left(\underline{u}\,\underline{v} + \underline{v}\,\underline{u}\right) - d(\underline{u}\,\underline{\omega} + \underline{\omega}\,\underline{u})$

Type-N: $\underline{\overline{\psi}}_0 = \underline{\overline{\psi}}_1 = \underline{\overline{\psi}}_2 = \underline{\overline{\psi}}_3 = 0$ (principal null direction of $C_{\lambda\mu rp}$ are $l^\mu, l^\mu, l^\mu, l^\mu$).

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