Some Basic Results on Nonsmooth Cases in Nonlinear Programming Problems: A Review

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Abstract - In the present paper, we have reviewed various works related to nonsmooth cases in nonlinear programming problems. We have studied comparative research in the said field.

Keywords: Linear programming, nonlinear programming, nonsmooth cases.

I. INTRODUCTION

Linear Programming deals with the optimization (maximization or minimization) of a linear function of variables known as objective function, subject to a set of linear equations and / or inequalities known as constraints. Once a problem is formulated as mathematical model, the next step is to solve the problem to get the optimal solution. A linear programming problem with only two variables can be solved by graphical method. In case of more than two variables, graphical method can't be applied. It was in 1947 that George Dantzig and his associates found out a technique for solving military planning problems while they were working on a project for U.S. Air Force. This technique consisted of representing the various activities of an organization as a linear programming model and arriving at the optimal programme by minimizing a linear objective function. Afterwards, Dantzig suggested this approach for solving business and industrial problems. He also developed the most powerful mathematical tool known as "simplex method" to solve linear programming problems.

In a nonlinear programming problem either the objective function and/or one or more of the constraints are nonlinear. A continuously differentiable function is called smooth function. A function is said to be of class C^k if the derivatives up to k^{th} order exist and are continuous and is called k^{th} order smoothness.

Nonsmoothfunctions include nondifferentiable and discontinuous functions. Graphs of non-differentiable functions may have abrupt bends.

Nonsmooth optimization (NSO) refers to the general problem of minimizing (or maximizing) functions that are typically not differentiable at their minimizers (maximizers). Since the classical theory of optimization presumes certain differentiability and strong regularity assumptions upon the functions to be optimized, it can not be directly utilized. However, due to the complexity of the real world, functions involved in practical applications are often non-smooth. That is, they are not necessarily differentiable. NSO problems arise in fields of image denoising, optimal control, neural network training, data mining, economics, computational chemistry and physics.

Moreover, using certain important methodologies for solving difficult smooth (continuously differentiable) problems leads directly to the need to solve non-smooth problems, which are either smaller in dimension or simpler in structure. This is the case, for instance in decompositions, dual formulations and exact penalty functions.

Finally, there exist so called stiff problems that are analytically smooth but numerically non-smooth. This means that the gradient varies too rapidly and, thus, these problems behave like non-smooth problems.

There are several approaches to solve NSO problems. The direct application of *smooth gradient-based methods* to non-smooth problems is a simple approach but it may lead to a failure in convergence, in optimality conditions, or in gradient approximation. All these difficulties arise from the fact that the objective function fails to have a derivative for some values of the variables.

II. PRELIMINARIES

Before starting the review work, we introduce some basicdefinitions related to the topic.

Definition 2.1. Convex set: A subset $C$of$R^n$is called convex if

$$\alpha x + (1-\alpha)y \in C \ \ \forall x, y \in C \in, \forall \alpha \in [0,1]$$

A subset CofR^n is convex if and only if for each integer $m \geq 1$,every convex combination of m points (m −simplex) of C is in C.

1. ∞n=0, such that the approximation xn+is determined as the poinof intersection (in the x-y plane) of the straight line through the points (xHessian **matrix :** It is matrix of second derivatives of the function f(x_1, x_2, x_3, x_4, ……………., xH(x) = $[a_{ij}]_{m \times n}$ where i = 1, 2, 3, …….., m j = 1, 2, 3, ………, **Convex**

set : A subset C of R^n is called convex ifConvex **set :** A subset C of $R^n$is called convex if $\alpha x + (1-\alpha)y \in C$ $\forall x, y \in C, \forall \alpha \in [0,$

Definition 2.2. Half space: Let $x \in R^n$, $x \neq 0$ and $\alpha \in R$. Then the set $\{y: y \in R^n,\ xy < \alpha\}$ is an open half space in R^n and the set $\{y: y \in R^n,\ xy \leq \alpha\}$ is a closed half space in R^n. Both half spaces are convex sets.

Definition 2.3. Plane: Let $x \in R^n$, $x \neq 0$ and $\alpha \in R$. Then the set

$\{y: y \in R^n,\ xy = \alpha\}$ is called a plane in R^n. Each plane in R^n is a convex set.

Definition 2.4. Subspace: A subset C of R^n is a subspace if $p_1x_1 + p_2x_2 \in C$, where $x_1, x_2 \in C$ and$p_1, p_2 \in R$. Clearly $0 \in C$ and C is a convex set.

Definition 2.5. Polytope: A set in R^n which is the intersection of a finite number of closed half spaces in R^n is called a polytope. If a polytope is bounded (i.e.,$\|x\| \leq \alpha$ for some fixed $\alpha \in R$), then it is called polyhedron. Polytopes and polyhedral are convex sets.

Definition 2.6.Convex combination : A point $x \in R^n$ is said to be a convex combination of the vectors x_1, x_2, $x_3, \dots\dots, x_n$in R^n if there exist n real numbers α_1, α_2, $\alpha_3, \dots\dots, \alpha_n$such that $x = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \cdots + \alpha_nx_n$

$\alpha_1,\ \alpha_2, \alpha_3, \dots\dots, \alpha_n \geq 0, \quad \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = 1$

Definition 2.7. Convex hull: Let C is a subset of R^n. The convex hull of C, denoted by $[C]$, is the intersection of all convex sets in R^ncontaining C.

The convex hull of any subset C of $R^n$is convex.

Also if C is convex then $C = [C]$.

Definition 2.8. (Hessian matrix): It is matrix of second derivatives of the function $f(x_1, x_2, x_3, \dots\dots\dots, x_m)$

$$H(x) = [a_{ij}]$$

Where $a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$

$$i = 1, 2, 3, 4, \dots\dots.., m j = 1, 2, 3, 4, \dots\dots.., n$$

Definition 2.9. Convex function: Let C be a convex subset of R^n. A function $f: C \to R^n$ is convex if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \quad \forall x, y \in C, \forall \alpha \in [0,1]$$

If the inequality $\leq$ is replaced by $<$then the function is called strictly convex function.

Definition 2.10. Concave function: Let C be a convex subset of R^n. A function $f: C \to R^n$ is concave ifthe function $(-f)$ is convex.

i.e. $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y),\ \forall x, y \in C, \forall \alpha \in [0,1]$

If the inequality $\geq$ is replaced by $>$then the function is called strictly concave function.

Definition2.11. (Sub gradient): A vector$g \in R^n$ is a sub gradient of

$f: R^n \to R$, at $x^* \in dom\ f$,

if for all $x \in dom\ f$,

$f(x) \geq f(x^*) + g^T(x - x^*)$.

If f is convex and differentiable, then its gradient at x^* is a subgradient. A sub gradient can exist even when f is not differentiable at $x^*$and there can be more than one sub gradient of a function f at a pointx^*. There are several ways to interpret a sub gradient. A vector g is a subgradient of f at x^*if the affine function (of x), $f(x^*) + g^T(x - x^*)$is a global under estimator of f. Geometrically, $g$is a sub gradient of f at x^* if $(g, -1)$ supports $epi\ f$at$(x^*, f(x^*))$. A function f is called subdifferentiable at x^*if there exists at least one sub gradient at x^*. The set of subgradients of f at the point $x^*$is called the subdifferential of f at x^*, and is denoted $\partial f(x^*)$. A function f is called subdifferentiable if it is subdifferentiable at all $x^* \in dom\ f$.

Example 2.1. Consider$f(x) = |x|$. For $x < 0$, the subgradient is unique: $\partial f(x) = \{-1\}$. Similarly, for $x > 0$ we have $\partial f(x) = \{1\}$. At $x = 0$ the subdifferential is defined by the inequality $|x| \geq gx$ for all x, which is satisfied if and only if $g \in [-1,1]$. Therefore, we have $\partial f(0) = [-1,1]$. The subdifferential $\partial f(x)$is always a closed convex set, even if $f$is not convex. In addition, if f is continuous at x, then the subdifferential $\partial f(x)$ is bounded.

Definition 2.12. (Lagrangian function) NLPP with equality constraints:

A general NLPP having m variables and n equality constraints $(m \geq n)$can be expressed as

Maximize or minimize $z = f(x)$

subject to constraints $g^j(x) = b_j,\ j = 1, 2, 3, 4, \dots, n$

where $x = (x_1, x_2, x_3, \dots, x_m) \geq 0,\ i = 1, 2, 3, 4, \dots, m$

the linear constraints can also be written as

$h^j(x) = g^j(x) - b_j,\ j = 1, 2, 3, 4, \dots, n$

The Lagrangian function is formed as

$$L(x, \lambda) = f(x) - \sum \lambda_j h^j(x),$$

where $\lambda = (\lambda_1,\ \lambda_2,\ \lambda_3, \dots\dots, \lambda_m)$ is Lagrange Multiplier.

Assuming that all the function are differentiable.

The necessary conditions for the objective function to be a maximum or a minimum are

$$\frac{\partial L}{\partial x_i} = 0 \;\; or \;\; \frac{\partial f}{\partial x_i} = \sum \lambda_j h_i^j(x)$$

and $\frac{\partial L}{\partial \lambda_j} = 0 \, or h^j(x) = 0, \quad i = 1,2,3,4,\dots,mj = 1,2,3,4,\dots,n$

The above necessary conditions also become the sufficient conditions for a maximum if the objective function is concave and for a minimum if the objective function is convex.

NLPP with inequality constraints :

Consider the following NLPP

Maximize $z = f(x)$

subject to $g^j(x) \leq b_j$

where $x = (x_1, x_2, x_3, \dots, x_m) \geq 0,\; i = 1,2,3,4,\dots,m$

The constraint equation can be written in the form

$h^j(x) = g^j(x) - b_j \leq 0,\; j = 1,2,3,4,\dots,n$

which can be further modified to equality constraint by introducing slack variables.

$$h^j(x) + s_j^2 = 0$$

The Lagrangian function is formed as

$$L(x,s,\lambda) = f(x) - \sum \lambda_j \, [h^j(x) + s_j^2]$$

The necessary conditions are

$$\frac{\partial L}{\partial x_i} = 0 \;\; or \;\; \frac{\partial f}{\partial x_i} = \sum \lambda_j h_i^j(x)$$

$$\frac{\partial L}{\partial x_i} = 0 \;\; or \;\; f_i(x) - \sum \lambda_j h_i^j(x) = 0$$

$$\frac{\partial L}{\partial \lambda_j} = 0 \, or h^j(x) + s_j^2 = 0$$

$$\frac{\partial L}{\partial s_j} = 0 \;\; or \; s_j \lambda_j = 0$$

The above conditions can be replaced by following conditions known as Kuhn – Tucker conditions.

$f_i(x) - \sum \lambda_j \, h_i^j(x) = 0, \qquad \lambda_j h^j(x) = 0,$

$h^j(x) \leq 0, \lambda_j \geq 0, i = 1,2,3,4,\dots,mj = 1,2,3,4,\dots,n$

The Kuhn – Tucker conditions are also the sufficient conditions for a maximum if the objective function $f(x)$is concave and all $h^j(x)$ are convex in x.

The Kuhn – Tucker conditions for a minimization type NLPP are

$f_i(x) - \sum \lambda_j \, h_i^j(x) = 0,$

$\lambda_j h^j(x) = 0,$

$h^j(x) \geq 0,$

$\lambda_j \geq 0, i = 1,2,3,4,\dots,mj = 1,2,3,4,\dots,n$

The Kuhn-Tucker conditions are also the sufficient conditions for a minimum if the objective function $f(x)$ is convex and all $h^j(x)$ are concave in x.

Definition 2.13. Secant Method: An effective iterative method used for solving $f(x) = 0$is the secant method. This method is derived by a linear interpolation procedure as follows:

Starting with two initial approximations $x_0$and $x_1$to the solution of $f(x) = 0$, we compute a sequence of approximations $\{x_n\}$, such that the approximation $x_{n+1}$is determined as the point of intersection(in the $x - yplane$) of the straight line through the points $\left(x_n , f(x_n)\right)$and $\left(x_{n-1}, f(x_{n-1})\right)$withthe$x$-axis. Since the equation of this straight line is

$$y = f(x_n) + \frac{\{f(x_n) - f(x_{n-1})\}}{\{x_n - x_{n-1}\}}(x - x_n)$$

In terms of divided differences, it can be written in the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, \; x_{n-1}]}$$

Definition 2.14.A differentiable numerical function θ defined on a set $C \subset R^n$is said to be η-convex at $x^* \in C$ if there exists a function $\eta(x, x^*)$ defined on $C \times C$ such that

$\theta(x) - \theta(x^*) \geq \eta'(x, x^*) \, \nabla\theta(x^*))$ for all $x \in C$.

θ is said to be η-convex on C if there exists a function $\eta(x_1, x_2)$ defined on $C \times C$ such that

$\theta(x_1) - \theta(x_2) \geq \eta'(x_1, x_2) \, \nabla\theta(x_2)$ for all $x_1, x_2 \in C$.

If we have strict inequality then θ is said to be η-strictly convex at x^* and η-strictly convex on C respectively.

Definition 2.15.A differentiable numerical function θ defined on a set$C \subset R^n$is said to be η-quasiconvex at $x^* \in C$ if there exists a function $\eta(x, x^*)$ defined on $C \times C$ such that

$\theta(x) \leq \theta(x^*) \Rightarrow \eta'(x, x^*) \, \nabla\theta(x^*) \leq 0$ for all $x \in C$.

θ is said to be η-quasiconvex on C if there exists a function $\eta(x_1, x_2)$ defined on $C \times C$ such that

$\theta(x_1) \leq \theta(x_2) \Rightarrow \eta'(x_1, x_2) \, \nabla\theta(x_2) \leq 0$ for all $x_1, x_2 \in C$.

Definition 2.16.A differentiable numerical function θ defined on a set $C\subset R^n$is said to be η-pseudoconvex at $x^* \in C$ if there exists a function $\eta(x, x^*)$ defined on $C \times C$ such that

$\eta'(x, x^*)\, \nabla\theta(x^*) \geq 0 \;\Rightarrow \theta(x) \geq \theta(x^*)$for all $x \in C$.

θ is said to be η-pseudoconvex on C if there exists a function $\eta(x_1, x_2)$ defined on $C \times C$ such that

$\eta'(x_1, x_2)\, \nabla\theta(x_2) \geq 0 \;\Rightarrow \theta(x_1) \geq \theta(x_2)$for all $x_1, x_2 \in C$.

Definition 2.17. An $m-$dimensional vector function $(g_1, g_2, g_3, \dots\dots, g_n)$defined on a set $C\subset R^n$is said to be η-convex, η-strictly convex, η-quasiconvex, η-pseudo convex at at$x^* \in C\ (on\ C)$if each g_i is η-convex, η-strictly convex, η-quasiconvex, η-pseudo convex at $x^* \in C\ (on\ C)$.

Definition 2.18. Quasiconvex function : A numerical functionθ defined on a set $C\subset R^n$is said to be quasiconvex at $x^* \in C$(with respect to C) if for each$x\in C$such that$\theta(x) \leq \theta(x^*)$, the function θ assumes a value no larger than $\theta(x^*)$on each point in the intersection of the closed line segment$[x^*, x]$and C, or equivalently

$$x \in C,\ \ \theta(x) \leq \theta(x^*),\ \ 0 \leq \lambda \leq 1,\ \ (1-\lambda)x^* + \lambda x \in C$$

$$\Rightarrow\ \theta[(1-\lambda)x^* + \lambda x] \leq\ \theta(x^*)$$

θ is said to be quasiconvex on C if it is quasiconvex at each $x \in C$.

Anumerical function θ defined on a convex set C is quasiconvex on C if and only if

$$x_1, x_2 \in C, \qquad \theta(x_2) \leq \theta(x_1), \qquad 0 \leq \lambda \leq 1$$

$\Rightarrow\ \theta[(1-\lambda)x_1 + \lambda x_2] \leq \theta(x_1)$

Definition 2.19. Quasiconcave function: A numerical functionθ defined on a set $C\subset R^n$is said to be quasiconcave at $x^* \in C$(with respect to C) if for each$x\in C$such that $\theta(x) \geq \theta(x^*)$, the function θ assumes a value no smaller than $\theta(x^*)$on each point in the intersection of the closed line segment$[x^*, x]$and C, or equivalently

$$x \in C,\ \ \theta(x^*) \leq \theta(x),\ \ 0 \leq \lambda \leq 1,\ \ (1-\lambda)x^* + \lambda x \in C$$

$$\Rightarrow\ \theta(x^*) \leq \theta[(1-\lambda)x^* + \lambda x]$$

θ is said to be quasiconcave on C if it is quasiconcave at each $x \in C$.

Obviously θ is quasiconcave at$x^*[on\ C]$if and only if $(-\theta)$ is quasiconvex at$x^*[on\ C]$.

Anumerical function θ defined on a convex set C is quasiconcave on C if and only if

$$x_1, x_2 \in C, \qquad \theta(x_1) \leq \theta(x_2), \qquad 0 \leq \lambda \leq 1$$

$\Rightarrow\theta(x_1) \leq \theta[(1-\lambda)x_1 + \lambda x_2]$

Definition 2.20. Pseudoconvex function : Let θ be a numerical function defined on some open set inR^n containing C. θ is said to be pseudo convex at $x^* \in C$(with respect to C) if it is differentiable at$x^*$and

$x \in C,\ \nabla\theta(x^*)(x - x^*) \geq 0$

$\Rightarrow \theta(x) \geq \theta(x^*)$

θ is said to be pseudo convex on Cif it is pseudo convex at each $x \in C$.

Definition 2.21. Pseudo concave function: Let θ be a numerical function defined on some open set inR^n containing C. θ is said to be pseudo concave at $x^* \in C$(with respect to C) if it is differentiable at$x^*$and

$x \in C,\ \nabla\theta(x^*)(x - x^*) \leq 0$

$\Rightarrow \theta(x) \leq \theta(x^*)$

θ is said to be pseudo concave on Cif it is pseudo concave at each $x \in C$.

Obviously θ is pseudoconcave atx^*[on C] if and only if $(-\theta)$ is pseudoconvex atx^* [on C].

Definition 2.22.α −invex set: A subset X of R^n is said to be an α −invex set, if there exist

$\eta: X \times X \to R^n, \alpha(x, u): X \times X \to R_+$ such that

$$u + \lambda\alpha(x, u)\eta(x, u) \in X, \forall x, u \in X, \lambda \in [0,1]$$

If$\alpha(x, u) = 1$,then$\alpha-$invex set becomes the invex set. It is well known that the α −invex set may not be convex sets.

Definition 2.23.α −preinvex function: The function f on the$\alpha-$invex set is said to beα −preinvex function**,** if there exist$\eta: X \times X \to R^n,\ \alpha(x, u): X \times X \to R_+$ such that

$$f\{u + \lambda\alpha(x, u)\eta(x, u)\} \leq (1-\lambda)f(u) + \lambda f(x), \forall x, u \in X, \qquad \lambda \in [0,1]$$

Definition 2.24. A point $x^* \in D$is said to be a weak Pareto efficient solution for (P)if the relation

$f(x) \nless f(x^*)$holds for all$x \in D$.

Definition 2.25.A point $x^* \in D$is said to be a locally weak Pareto efficient solution for(P)if there is a neighborhood $N(x^*)$around x^*such that

$f(x) \nless f(x^*)$holds for all $x \in N(x^*) \cap D$.

Definition 2.26. The function g is said to satisfy the generalized Slater's constraint qualification at$x^* \in D$ if g is d −invex at $x^*$and there exists $x' \in D$such that $g_j(x') < 0, j \in J(x^*)$.

Definition 2.27.A function $f: X \to R^k$be defined on X and directionally differentiable at $u \in X$ is said to be $\alpha - d - invex$at $u \in X$with respect to η if for any $x \in X$,

$f(x) - f(u) \geq \alpha(x,u) f'\{u, \eta(x,u)\}$

Definition 2.28. A mapping $h: E \to G$ is said to be strongly compact Lipschitzian at $\bar{x} \in E$ if there exist a multifunction $R: E \to Comp\ G$ [$Comp\ G =$ the set of all norm compact subsets of G] and a function $r: E \times E \to R_+$ satisfying the following conditions:

(i) $\lim_{d \to 0, x \to \bar{x}} r(x,d) = 0$

(ii) There exists $\alpha > 0$ such that

$$t^{-1}[h(x+td) - h(x)] \in R(d) + \|d\| r(x,t) B_G$$

Definition 2.29.A normed space B is called Banach space if every Cauchy sequence in B is convergent.

III. MAIN RESULTS (REVIEWED)

3.1 K. Ritter [1] introduced the duality for non-linear programming in a Banach space.

A duality theorem provides a relationship between a constrained maximization type problem and a constrained minimization type problem in non-linear programming. The maximization type problem is called the primal and the other the dual. Primal and dual are related in such a way that solution exists for both problems. If a solution exists then optimal function values for the primal and dual problems are equal.

Many results for completely non-linear programming problems are obtained by P. Wolfe [2], M. A. Hanson [3], P. Huard [4] and 0. L. Mangasarian [5]. The results for the infinite-dimensional case have been obtained by M. A. Hanson [6] who proved a duality theorem which is based on Hurwicz's generalization [7] of the Kuhn-Tucker theorem to locally convex spaces.

K. Ritter [1] considered the non-linear duality problem in a real Banach space.

Let u and v are arbitrary elements of in a real Banach space $B$and B^* denotes the conjugate space of continuous linear functionals over B^*.

Let $g(u)$ be a Frechet's differentiable functional over B and $h_j(u)$ are Frechet's differentiable functional. The derivatives $g'(u)$ and $h_j'(u)$ belong to B^*.

Primal problem:

Maximize $g(u)$subject to the constraints

$$h_j(u) \leqq 0, \quad j = 1, 2, 3, \ldots, n$$

Dual problem:

Minimize $g(u) - \sum_{j=1}^{n} \lambda_j h_j(u)$ subject to the constraints

$$g'(u) - \sum_{j=1}^{n} \lambda_j h_j'(u) = 0,$$

$$\lambda_j \geqq 0, \quad j = 1, 2, 3, \ldots, n.$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real scalars.

The following theorem of Kuhn et al.[8] is needed in the proof of duality theorem [1].

Theorem 3.1.1.(i) If $g(u)$ and $h_j(u)$, $j = 1, 2, 3, \ldots, n$, are Frechet's differentiable then the following conditions are necessary for u_0 to be a local maximum of $g(u)$ subject to the constraints $h_j(u) \leqq 0, \quad j = 1, 2, 3, \ldots, n$:

$$(a) g'(u_0) = \sum_{j=1}^{n} \lambda_j^0 h_j'(u_0)$$

$$(b) \lambda_j^0 h_j(u_0) = 0, \qquad j = 1, 2, 3, \ldots, n$$

$$(c) \lambda_j^0 \geqq 0, \qquad j = 1, 2, 3, \ldots, n$$

(ii) If, in addition, $g(u)$ is concave and all $h_j(u)$ are convex then above conditions (a), (b) and (c) are also sufficient conditions for u_0 to be the absolute maximum of $g(u)$ subject to the constraints $h_j(u) \leqq 0, \quad j = 1, 2, 3, \ldots, n$.

The following lemma of Wolfe [2] is needed in the proof of duality theorem [1].

Lemma 3.1.1. Let $g(u)$ is a concave and differentiable functional over $B$and all $h_j(u)$ are convex and differentiable functionals over B.

(i) If u_0 is any feasible solution of the primal problem and $(u_1, \lambda_1^1, \lambda_2^1, \ldots, \lambda_n^1)$ is any feasible solution of the dual problem, then

$$g(u_0) \leqq g(u_1) - \sum_{j=1}^{n} \lambda_j^1 h_j(u_1)$$

(ii) If $g(u_0) = g(u_1) - \sum_{j=1}^{n} \lambda_j^1 h_j(u_1)$ holds, then u_0 is an optimal solution of the primal problem and $(u_1, \lambda_1^1, \lambda_2^1, \ldots, \lambda_n^1)$ is an optimal solution of the dual problem.

Theorem 3.1.2. $(\textbf{Duality theorem})$. Let $g(u)$ is a concave and differentiable functional over $B$and all $h_j(u)$ are convex and differentiable functionals over B.

(i) If u_0 is an optimal solution of the primal problem then there exists scalars $\lambda_1^0, \lambda_2^0, \ldots, \lambda_n^0$ such that $(u_0, \lambda_1^0, \lambda_2^0, \ldots, \lambda_n^0)$ is an optimal solution of the dual problem, and the extreme values are equal.

(ii) If $(u_0, \lambda_1^0, \lambda_2^0, \ldots, \lambda_n^0)$ is an optimal solution of the dual problem then u_0 is an optimal solution of the primal problem, and the extreme values are equal.

Lemma 3.1.2. A feasible solution $(u_1, \lambda_1^1, \lambda_2^1, \ldots, \lambda_n^1)$ is an optimal solution of the dual problem if and only if

$$h_j(u_1) \leqq 0, \quad \lambda_j^1 h_j(u_1) = 0, \qquad j = 1, 2, 3, \ldots, n.$$

3.2. R.N. Kaul et al. [9] introduceda new class of functions, called η-convex, η-quasiconvex and η-pseudoconvex and obtained the following interrelations between these functions.

Theorem 3.2.1. Every differentiable convex function is η-convex but the converse is not true.

Theorem 3.2.2. Every differentiable strictly convex function is η-strictly convex but the converse is not true.

Theorem 3.2.3. Every differentiable quasiconvex function is η-quasiconvex but the converse is not true.

Theorem 3.2.4. Every pseudo convex function is η-pseudoconvex but the converse is not true.

Theorem 3.2.5. Everyη-convex function is η-quasiconvex for the same function η but the converse is not true.

Theorem 3.2.6. Every η-convex function isη-pseudoconvex for the same function η but the converse is not true.

Theorem 3.2.7. Every strictly η-convex function is η-convex for the same function η but the converse is not true.

R.N. Kaul et al. [9] obtained a number of sufficient optimality criteria for non-linear programming problems involving classes of η-convex, η-quasiconvex and η-pseudoconvex functions.

Sufficient Optimality Criteria:

Let θ be a numerical function and g be an $m-$dimensional vector function defined on $C \subset R^n$. Consider the non-linear programming problem

Minimize $\theta(x)$, subject to $g(x) \leq 0,\ x \in C$. (MP)

Let $X = \{x \in C : g(x) \leq 0\}$ denote the set of all feasible solutions of (MP). In this section, a number of sufficient optimality criteria are obtained in the form of the following theorems which do not depend on the convexity of the functions involved.

Theorem 3.2.8. Let $x^* \in C$ and let θ and g be η-convex at x^* for the same function η . If there exist $u_0^* \in R$ and $u^* \in R^m$ such that (x^*, u_0^*, u^*) satisfies the following conditions:

$$\nabla\left(u_0^*\theta(x^*)\right) + \nabla\left(u^{*'}g(x^*)\right) = 0 \qquad (1)$$

$$g(x^*) \leq 0, \qquad (2)$$

$$u^{*'}g(x^*) = 0, \qquad (3)$$

$$(u_0^*, u^*) \geq 0, \ (u_0^*, \quad u^*) \neq 0, \qquad (4)$$

$$u_0^* > 0$$

then x^* is an optimal solution of (MP).

The sufficiency Theorem 2.1 of Hanson [10] now follows from the above theorem 3.2.8 and can be restated as follows.

Corollary3.2.1. Let $x^* \in C$ and let θ and g be η-convex at x^* for the same function η . If there exists$u^* \in R^m$ such that (x^*, u^*) satisfies the following conditions:

$$\nabla\left(\theta(x^*)\right) + \nabla\left(u^{*'}g(x^*)\right) = 0 \qquad g(x^*) \leq 0,$$

$$u^{*'}g(x^*) = 0, \qquad u^* \geq 0$$

then x^* is an optimal solution of (MP).

Theorem 3.2.9. Let $x^* \in C$ and let θ be η-convex at x^* and g be η-strictly convex at x^* for the same function η. If there exist $u_0^* \in R$ and $u^* \in R^m$ such that (x^*, u_0^*, u^*) satisfies (1) - (4) of theorem 3.1.8, then x^* is an optimal solution of (MP).

Mangasarian [11] has speculated that pseudo convexity of θ and quasiconvexity of g are the weakest conditions that can be imposed so that the conditions of corollary 3.2.1 are sufficient optimality. In the next theorem R.N. Kaul et al. [9] have shown that these conditions are also sufficient for optimality when θ is η-pseudoconvex and g isη-quasiconvex.

Theorem 3.2.10. Let $x^* \in C$ and let= $\{i : g_i(x^*) = 0\}$. Let θ beη-pseudoconvex at x^* and $g be\, \eta -$quasiconvex at x^* for the same function η . If$u^* \in R^m$ such that (x^*, u^*) satisfies the conditions of corollary 3.2.1, then x^* is an optimal solution of (MP).

3.3.N. G. Rueda et al. [12] introduced optimality criteria for Type I and Type II functions. The concept of invexity was introduced by Hanson [10] as a generalization of convexity for constrained optimization problems of the form

min $f(x)$ for $x \in X \subseteq R^n$

subject to $g(x) \leq 0$ …….. (1.1)

where$f : X \to R$ and $g : X \to R^m$ are differentiable functions on a set $X \subseteq R^n$. Hanson [10] showed that weak duality and sufficiency of the Kuhn Tucker conditions hold when invexity is required instead of the usual requirement of convexity.

Subsequently, Hanson and Mond introduced two new classes of functions which are not only sufficient but are also necessary for optimality in primal and dual problems, respectively. Let $P = \{\, x : x \in X, g(x) \leq 0 \,\}$

and $D = \{ x: (x, y) \in Y \}$

where $Y = \{(x, y) : x \in X, \ y \in R^m, \nabla_x f(x) + y'[\nabla_x g(x)] = 0, y \geq 0 \}$.

Hanson and Mond defined $f(x)$ and $g(x)$ as Type I objective and constraint functions, respectively, with respect to $\eta(x)$ at x_0 if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that

$$f(x) - f(x_0) \geq [\nabla_x f(x_0)]' \eta(x)$$

and

$$-g(x_0) \geq [\nabla_x g(x_0)] \, \eta(x)$$

and $f(x)$ and $g(x)$ as Type II objective and constraint functions, respectively, with respect to $\eta(x)$ at x_0 if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in D$ such that

$$f(x_0) - f(x) \geq [\nabla_x f(x)]' \eta(x)$$

and

$$-g(x) \geq [\nabla_x g(x)] \, \eta(x)$$

In the definitions of Type I and Type II functions we shall consider x_0 to be fixed. If x_0 is not fixed, the definition of Type I would be written as

$$f(x) - f(x_0) \geq [\nabla_x f(x_0)]' \eta(x, x_0)$$

and

$$-g(x_0) \geq [\nabla_x g(x_0)] \, \eta(x, x_0)$$

and the definition of Type II would be equivalent to this.

The following two theorems are useful to find sufficient conditions for Type I and Type II functions.

Theorem 3.3.1. If $f(x)$ and $g(x)$ are differentiable at x_0 and there exists an n-dimensional vector function $\eta(x)$ such that

$$f\big(x_0 + \lambda\eta(x)\big) \leq \lambda f(x) + (1-\lambda) f(x_0), \qquad 0 \leq \lambda \leq 1,$$

and

$$g\big(x_0 + \beta\eta(x)\big) \leq (1-\beta) g(x_0), \qquad 0 \leq \beta \leq 1$$

for all $x \in P$, then $f(x)$ and $g(x)$ are Type I.

Theorem 3.3.2. If $f(x)$ and $g(x)$ are differentiable on D and there exists an n-dimensional vector function $\eta(x)$ such that

$$f\big(x + \lambda\eta(x)\big) \leq \lambda f(x_0) + (1-\lambda) f(x), \qquad 0 \leq \lambda \leq 1,$$

and

$$g\big(x + \beta\eta(x)\big) \leq (1-\beta) g(x), \qquad 0 \leq \beta \leq 1$$

for all $x \in D \ at \ x_0$, then $f(x)$ and $g(x)$ are Type II.

Some of the results obtained by R. N. Kaul et al. [9] can be adapted to Type I and Type II functions.

Theorem 3.3.3. If $f(x)$ and $g(x)$ are convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are Type I.

Theorem 3.3.4. If $f(x)$ and $g(x)$ are convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are Type II.

Theorem 3.3.5. If $f(x)$ and $g(x)$ are strictly convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are strictly Type I.

Theorem 3.3.6. If $f(x)$ and $g(x)$ are strictly convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are strictly Type II.

Theorem 3.3.7. If $f(x)$ and $g(x)$ are Type I objective and constraint functions, respectively, with respect to a common $\eta(x)$ at x_0, then $f(x)$ and $g(x)$ are pseudo-Type I for the same $\eta(x)$.

Theorem 3.3.8. If $f(x)$ and $g(x)$ are Type II objective and constraint functions, respectively, with respect to a common $\eta(x)$ at x_0, then $f(x)$ and $g(x)$ are pseudo-Type II for the same $\eta(x)$.

Theorem 3.3.9. If $f(x)$ and $g(x)$ are Type I objective and constraint functions, respectively, with respect to a common $\eta(x)$ at x_0, then $f(x)$ and $g(x)$ are quasi-Type I for the same $\eta(x)$.

Theorem 3.3.10. If $f(x)$ and $g(x)$ are Type II objective and constraint functions, respectively, with respect to a common $\eta(x)$ at x_0, then $f(x)$ and $g(x)$ are quasi-Type II for the same $\eta(x)$.

Theorem 3.3.11. If $f(x)$ and $g(x)$ are strictly Type I objective and constraint functions, respectively, with respect to a common $\eta(x)$ at x_0, then $f(x)$ and $g(x)$ are Type I.

Theorem 3.3.12. If $f(x)$ and $g(x)$ are strictly Type II objective and constraint functions, respectively, with respect to a common $\eta(x)$ at x_0, then $f(x)$ and $g(x)$ are Type II.

R. N. Kaul et al. [9] considered a number of sufficient optimality criteria which do not depend on convexity. Those results can be adapted to the class of Type I functions.

Theorem 3.3.13. Let $x^* \in X \ and f(x)$ and $g(x)$ be Type I objective and constraint functions, respectively, with respect to a common $\eta(x)$ at x^*. If there exist $\mu_0^* \in R$ and $\mu^* \in R^m$ such that (x^*, μ_0^*, μ^*) satisfies the following conditions.

$$\nabla_x\big(\mu_0^* f(x^*)\big) + \nabla_x\left(\mu^{*'} g(x^*)\right) = 0 \qquad (1)$$

$$g(x^*) \leq 0 \qquad (2)$$

$\mu^{*'} g(x^*) = 0$ (3)

$(\mu_0^*, \mu^*) \geq 0,\ (\mu_0^*, \mu^*) \neq 0$ (4)

$\mu_0^* > 0$ (5)

then x^* is an optimal solution of (1.1).

Corollary 3.3.1. Let $x^* \in X\ and f(x)$ and $g(x)$ be Type I objective and constraint functions, respectively, with respect to a common $\eta(x)$ at x^*. If there exists $\mu^* \in R^m$ such that $(x^*,\ \mu^*)$ satisfies the following conditions.

$$\nabla_x f(x^*) + \nabla_x \left(\mu^{*'} g(x^*)\right) = 0 \quad (1)$$

$$g(x^*) \leq 0 \quad (2)$$

$$\mu^{*'} g(x^*) = 0 \quad (3)$$

$$\mu^* \geq 0 \quad (4)$$

then x^* is an optimal solution of (1.1).

Theorem 3.3.14. Let $x^* \in X\ and$

$f(x) - f(x^*) \geq [\nabla_x f(x^*)]' \eta(x)$ (1)

and $-g(x^*) > [\nabla_x g(x^*)] \eta(x)$ (2)

for the same $\eta(x)$. If there exist $\mu_0^* \in R$ and $\mu^* \in R^m$ such that $(x^*,\ \mu_0^*,\ \mu^*)$ satisfies (1) – (4) of the theorem 3.2.13., then x^* is an optimal solution of (1.1).

Theorem 3.3.15. Let $x^* \in X\ and$ let $I = \{i: g^i(x^*) = 0\}$. Let $f(x)$ satisfies $[\nabla_x f(x^*)]' \eta(x) \geq 0 \implies f(x) \geq f(x^*)$ and let g_i satisfies $-g_i(x^*) \leq 0 \implies [\nabla_x g_i(x^*)] \eta(x) \leq 0$ for the same $\eta(x)$. If there exists $\mu^* \in R^m$ such that $(x^*$, $\mu^*)$ satisfies conditions (1) – (4) of corollary 3.2.1, then x^* is an optimal solution of (1.1).

Theorem 3.3.16. Let $x^* \in X$. If there exists $\mu^* \in R^m$ such that $(x^*,\ \mu^*)$ satisfies conditions (1) – (4) of corollary 3.2.1 and if $f(x)$ satisfies$[\nabla_x f(x^*)]' \eta(x) \geq 0 \implies f(x) \geq f(x^*)\ and g_i(x)$ satisfies

$-\mu_i^{*'} g_i(x^*) \leq 0 \implies [\nabla_x \left(\mu_i^{*'} g_i(x^*)\right)] \eta(x) \leq 0$ for the same $\eta(x)$, then x^* is an optimal solution of (1.1).

3.4. S.K. Mishra et al. [13] considered a non-differentiable and multiobjective programming problem and derived some Karush–Kuhn–Tucker type of sufficient optimality conditionsfor a (weakly) Pareto efficient solution to the problem involving the new classes of directionallydifferentiable generalized type-I functions. Furthermore, the Mond–Weir type and general Mond–Weir type of duality results are also obtained in terms of right differentials of the aforesaid functions involved in the multi objective programming problem.

Consider the following multi-objective programming problem:

$$(P) \quad \min f(x)$$

such that $g(x) \leq 0, x \in X$,

where$f: X \to R^k$, $g: X \to R^m$, X is a non-empty open α −invex ofR^n, $\eta: X \times X \to R^n$ isa vector function.

The following results from Antczak [14] and Weir and Mond [15] are needed.

Lemma 3.4.1. If$\bar{x}$is a locally weak Pareto or weak Pareto efficient solution for(P)andg_j is continuous at $\bar{x}$for $j \in J'(\bar{x})$, then the following system of inequalities:

$$f'(\bar{x}, \eta(x, \overline{x})) < 0,\ g'_{J(\bar{x})}(\bar{x}, \quad \eta(x, \bar{x})) < 0$$

has no solution for $x \in X$.

Lemma 3.4.2. Let S be a non empty set inR^n and $\Psi: S \to R^p$be a preinvex function on S. Then either $\Psi(x) < 0$ has a solution $x \in S$, or $\lambda^T \Psi(x) \geqq 0$ for all $x \in S$, or some$\lambda \in R_+^m$,but both alternatives are never true.

Lemma 3.4.3.(F.John type necessary optimality condition):Let$\bar{x}$be a weak Pareto efficient solution for (P).Moreover, we assume that g_j is continuous for$j \in J'(\bar{x})$, f and g are directionally differentiable at$\bar{x}$with $f'(\bar{x},\ \eta(x, \bar{x}))$ and $g'_{J(\bar{x})}(\bar{x},\ \eta(x, \bar{x}))$pre-invex functions of x on X. Then there exist $\bar{\xi} \in R_+^k$, $\bar{\mu} \in R_+^m$ such that $(\bar{x}, \bar{\xi}, \bar{\mu})$ satisfies the following conditions:

$$\bar{\xi}^T f'(\bar{x}, \eta(x, \overline{x})) + \bar{\mu}^T g'(\bar{x}, \eta(x, \overline{x})) \geqq 0, \forall\ x \in X\ ,$$

$$\bar{\mu}^T g(\bar{x}) = 0,$$

$$g(\bar{x}) \leqq 0.$$

Lemma 3.4.4.(Karush-Kuhn-Tucker type necessary optimality condition)

Let $\bar{x}$ be a weak Pareto efficient solution for (P). Assume that g_j is continuous for$j \in J'(\bar{x})$, f and g are directionally differentiable at$\bar{x}$with $f'(\bar{x},\ \eta(x, \bar{x}))$ and $g'_{J(\bar{x})}(\bar{x},\ \eta(x, \bar{x}))$pre-invex functions of x on X. Moreover, we assume that $g satisfies$ the general Slater's constraint qualification at $\bar{x}$. Then there exists$\bar{\mu} \in R_+^m$ such that $(\bar{x}, \bar{\mu})$ satisfies the following conditions:

$$f'(\bar{x}, \eta(x, \overline{x})) + \bar{\mu}^T g'(\bar{x}, \eta(x, \overline{x})) \geqq 0, \forall\ x \in X\ , \quad (1)$$

$$\bar{\mu}^T g(\bar{x}) = 0, (2)$$

$$g(\bar{x}) \leqq 0.(3)$$

Sufficient optimality conditions

In this section S.K. Mishra et al. [13] established a Karush-Kuhn-Tucker type sufficient optimality condition.

Theorem 3.4.1. Let $\bar{x}$be a feasible solution for (P) at which conditions (1) - (3) are satisfied. Moreover, if any of the following conditions are satisfied:

(a) $(f_i,\sum_{j=1}^{m}\mu_j g_j)$is strong pseudo-quasi d-V–type –I at $\bar{x}$ with respect to $\eta, \alpha_i(x,u)$ and $\beta_j(x,u)$;

(b) $(f_i,\sum_{j=1}^{m}\mu_j g_j)$is weak strictly pseudo-quasi d–V–type–I at $\bar{x}$ with respect to $\eta, \alpha_i(x,u)$ and $\beta_j(x,u)$;

(c) $(f_i,\sum_{j=1}^{m}\mu_j g_j)$ is weak strictly pseudo-d-V–type–I at $\bar{x}$ with respect to $\eta, \alpha_i(x,u)$ and $\beta_j(x,u)$.

then $\bar{x}$ is a weak Pareto efficient solution for (P).

3.5. S.K. Mishra et al. [16] considered a vector optimization problem with functions defined on three Banach spaces,E, F and G.

Consider the following mathematical programming problem:

Min $\{f(x): x \in C, -g(x) \in K\}$where f and g are mappings from E into F and G respectively, and C and K are two subsets of E and G.

This problem has been investigated extensively in recent years. When F and G are finite-dimensional linear spaces, and$f$ and g are locally Lipschitz, problem $(P)$was studied by Clarke [17], Craven [18], Minami [19], Giorgi and Guerraggio [20], Reiland [21], Lee [22], Liu [23], Mishra and Mukherjee [24], Mishra [25], Kim [26] and Bhatia and Jain [27] among others. The Lipschitz infinite dimensional case was considered by El Abdouni and Thibault [28], Coladas, Li and Wang [29] and recently, by Brandão, Rojas-Medar and Silva [30]. Brandão, Rojas-Medar and Silva [30] studied multiobjective mathematical programming with non-differentiable strongly compact Lipschitz functions defined on general Banach spaces.

Under a Slater-type condition and an invexity notion for mappings defined between Banach spaces, Karush-Kuhn-Tucker type conditions and Mond-Weir type duality results are established in [30]. S.K. Mishra et al. [16] extended the concept of type-I functions [31], pseudo-type-I and quasi type-I functions [12], quasi-pseudo type-I, pseudo-quasi type-I [32] to the context of Banach spaces and established the sufficiency of Karush-Kuhn-Tucker type optimality conditions under weaker invexity assumptions than that of Bandão, Rojas-Medar and Silva[30].S.K. Mishra et al. [16] alsoobtained variousdualityresultsunderaforesaid assumptions andestablished sufficient optimality conditions.

Consider the following dual of problem (P):

maximize$f(w)(D)$

subject to : $w \in C, u^* \in Q^*, v^* \neq 0, v^* \in K^*$,

$\langle v^*, g(w)\rangle \geq 0, \qquad 0 \in \partial(u^* o\, f + v^* o\, g + k\partial_C)(w)$

S.K. Mishra et al. [16] provided weak and strong duality relations between Problems (P) and (D).

Theorem 3.5.1. (Weak Duality). Let x and (w, u^*, v^*)be feasible solutions for problems (P) and (D), respectively. Suppose that (f, g) are type-I at w with respect to$C$, for the sameη . Then,

$$f(x) < f(w).$$

Theorem 3.5.2. (Weak Duality). Let x and (w, u^*, v^*)be feasible solutions for problems (P) and (D), respectively. Suppose that (f, g) are pseudo-quasi-type-I at w with respect to$C$, for the sameη . Then,

$$f(x) < f(w).$$

Theorem 3.5.3. (Weak Duality). Let x and (w, u^*, v^*)be feasible solutions for problems (P) and (D), respectively. Suppose that (f, g) are quasi-strictly pseudo-type-I at w with respect to$C$, for the sameη . Then,

$$f(x) < f(w).$$

Theorem 3.5.4. (Strong Duality). Suppose that (f, g) are type-I at all feasible points x of (P), with respect to C, and assume that the restrictions of Problem (P) satisfy the Slater condition. If x_0 is a weak Pareto-optimal solution of (P), then there exists $(\bar{u}^*, \bar{v}^*) \in Q^* \times K^*$such that $\langle v^*, g(x_0)\rangle = 0, (x_0, \bar{u}^*, \bar{v}^*)$ is a weak Pareto-optimal solution for (D), and the objective values of the two problems are the same.

Theorem 3.5.5. (Strong Duality). Suppose that (f, g) are pseudo-quasi-type-I at all feasible points x of (P), with respect to C, and assume that the restrictions of Problem (P) satisfy the Slater condition. If x_0 is a weak Pareto-optimal solution of (P), then there exists $(\bar{u}^*, \bar{v}^*) \in Q^* \times K^*$such that

$\langle v^*, g(x_0)\rangle = 0, (x_0, \bar{u}^*, \bar{v}^*)$ is a weak Pareto-optimal solution for (D), and the objective values of the two problems are the same.

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