Common Fixed Point Theorem for A- Compatible Mappings Taking Integral Type Mappings

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Abstract: The aim of this paper is to generalize common fixed point theorem proved by Bijendra singh by introducing the two types of weak reciprocally continuous mappings. The concept of compatibility and complete metric space are replaced by two different types of weak reciprocally continuous mappings along with some weaker conditions for integral type. AMS: 47H10, 54H25.

Keywords: Fixed point, compatible mappings, weakly compatible mappings, associated sequence, reciprocally continuous mappings, S-compatible mapping, S - weak reciprocally continuous mappings.

1. INTRODUCTION

G. Jungck[1] introduced the concept of compatible maps which is weaker than weakly commuting maps. G.Jungck[1] generalized the concept of compatible mappings by introducing the notion of compatible mappings of type(A). Pathak extended the concept of compatibility to two analogous definitions namely A – compatible and S – compatible. After wards Jungck and Rhoades[4] defined weaker class of maps known as weakly compatible maps.

Pant[2] introduced a new notion of continuity namely reciprocal continuity for a pair of self maps and proved some common fixed point theorems. Further Pant[2] et al introduced the concept of weak reciprocally continuity.

Definition 1.1. Two self- maps A and S of a metric space (X,d) are said to be compatible mappings[1] if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.2. Two self- maps A and S of a metric space (X,d) are said to be weakly compatible mappings[4] if they commute at their coincidence point.

Definition 1.3. Two self- maps A and S of a metric space (X,d) are said to be reciprocally continuous [2] if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.4. Two self- maps A and S of a metric space (X,d) are said to be weak reciprocally continuous [11] if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.5. Two self- maps A and S of a metric space (X,d) are said to be A -weak reciprocally continuous iff $\lim_{n\to\infty} ASx_n = At$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.6. Two self- maps A and S of a metric space (X,d) are said to be S - weak reciprocally continuous iff $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.7. Two self- maps A and S of a metric space (X,d) are said to be A – compatible iff $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.8. Two self- maps A and S of a metric space (X,d) are said to be S – compatible iff $\lim_{n\to\infty} d(AAx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

It is clear that every compatible pair is weakly compatible but its converse need not be true.

Singh and Chauhan[6] proved the following theorem.

Theorem 1.9. Let A, B, S and T be self- mappings from a complete metric space (X,d) into itself satisfying the following conditions.

(i) A(X) ⊆ T(X) and B(X) ⊆ S(X),

(ii) The pairs {A, S} and {B, T} are compatible on X,

(iii) One of A, B, S and T is continuous,

(iv)

$$[d(Ax,By)]^2 \leq k_1[d(Ax,Sx)d(By,Ty)+d(By,Sx)d(Ax,Ty)]$$

$$+k_2[d(Ax,Sx)d(Ax,Ty)+d(By,Ty)d(By,Sx)]$$

Where $0 \leq k_1+2k_2 < 1$, $k_1, k_2 \geq 0$

Further if X is a complete metric space

Then A, B, S and T have a unique common fixed point in X.

Now we use definition of associated sequence[10] that plays a vital role in proving our theorem

Definition 1.10. Suppose A, B, S and T are self-maps of a metric space (X,d) satisfying the condition (i) Then for an arbitrary $x_0 \in X$ such that $Ax_0 = Tx_1$ and for this point x_1, there exist a point x_2 in X such that $Bx_1 = Sx_2$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_n\}$ in X such that

$y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n \geq 0$. We shall call this sequence as an " Associated sequence of x_0 " relative to the four self-maps A, B, S and T.

II. MAIN THEOREM

Theorem 2.1. Let A, B, S and T are self- maps of a metric space (X,d) satisfying:

(1) A(X) ⊆ T(X) and B(X) ⊆ S(X),

(2) $$\int_0^{[d(Ax,By)]^2} \emptyset(t)dt \leq k_1 \int_0^{d(Ax,Sx)d(By,Ty)+d(By,Sx)d(Ax,Ty)} \emptyset(t)dt$$

$$+ k_2 \int_0^{d(Ax,Sx)d(Ax,Ty)+d(By,Ty)d(By,Sx)} \emptyset(t)dt + k_3 \int_0^{d(Ax,Ty)\,d(By,Ty)} \emptyset(t)dt + k_4 \int_0^{d(Ax,Sx)d(Ax,By)} \emptyset(t)dt$$

for all elements x,y ∈ X.

(3) The pair (A,S) is A – weak reciprocally continuous and A – compatible or The pair (A,S) is S – weak reciprocally continuous and S – compatible

(4) The pair (B,T) is weakly compatible.

(5) For any $x_0 \in X$ the associated sequence relative to four self maps A, B, S and T such that the sequence Ax_0, Bx_1, Ax_2, Bx_3,….., Ax_{2n}, Bx_{2n+1},……. Converges to z ∈ X as $n \to \infty$.

also $\emptyset:[0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$, non-negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$ .Then A, B, S and T have a unique common fixed point in X.

Then A, B, S and T have a unique common fixed point z in X.

Proof. Using the condition (5) ,

We have $Ax_{2n} \to z$, $Tx_{2n+1} \to z$, $Bx_{2n+1} \to z$, $Sx_{2n} \to z$, as $n \to \infty$ (2.1.1)

Case 1:

Since S is weak reciprocally continuous then $\lim_{n\to\infty} SAx_{2n} \to Sz$

Since the pair (A,S) is S compatible then $\lim_{n\to\infty} d(SAx_{2n}, AAx_{2n}) = 0$

giving that $\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} AAx_{2n} = Sz$ (2.1.2)

Put $x = Ax_{2n}$, $y = x_{2n+1}$ in condition (2), we have

$$\int_0^{[d(AAx_{2n},Bx_{2n+1})]^2} \emptyset(t)dt \leq k_1 \int_0^{\left\{\begin{matrix} d(AAx_{2n},SAx_{2n})d(Bx_{2n+1},Tx_{2n+1})+ \\ d(Bx_{2n+1},SAx_{2n})d(AAx_{2n},Tx_{2n+1}) \end{matrix}\right\}} \emptyset(t)dt.$$

$$+ k_2 \int_0^{\left\{\begin{matrix} d(AAx_{2n},SAx_{2n})d(AAx_{2n},Tx_{2n+1})+ \\ d(Bx_{2n+1},Tx_{2n+1})d(Bx_{2n+1},SAx_{2n}) \end{matrix}\right\}} \emptyset(t)dt$$

$$+ k_3 \int_0^{d(AAx_{2n},Tx_{2n+1})d(Bx_{2n+1},Tx_{2n+1})} \emptyset(t)dt$$

$$+ k_4 \int_0^{d(AAx_{2n},SAx_{2n})d(AAx_{2n},Bx_{2n+1})} \emptyset(t)dt$$

Letting $n \to \infty$ on both sides and using (2.1.1),(2.1.2), then we get

$$\int_0^{[d(Sz,z)]^2} \emptyset(t)dt \leq k_1 \int_0^{\{d(Sz,Sz)d(z,z)+d(z,Sz)d(Sz,z)\}} \emptyset(t)dt.$$

$$+k_2 \int_0^{\{d(Sz,Sz)d(Sz,z)+d(z,z)d(z,Sz)\}} \emptyset(t)dt$$

$$+ k_3 \int_0^{d(Sz,z)d(z,z)} \emptyset(t)dt$$

$$+ k_4 \int_0^{d(Sz,Sz)d(Sz,z)} \emptyset(t)dt$$

$$\int_0^{[d(Sz,z)]^2} \emptyset(t)dt \leq k_1 \int_0^{[d(Sz,z)]^2} \emptyset(t)dt$$

We have Sz = z, since $0 \leq k_1 + k_2 + k_3 + k_4 < 1$, where $k_1, k_2, k_3, k_4 \geq 0$

Put x = z, y = x_{2n+1} in condition (2), we have

$$\int_0^{[d(Az,Bx_{2n+1})]^2} \emptyset(t)dt \leq k_1 \int_0^{\left\{\begin{matrix} d(Az,Sz)d(Bx_{2n+1},Tx_{2n+1})+ \\ d(Bx_{2n+1},Sz)d(Az,Tx_{2n+1}) \end{matrix}\right\}} \emptyset(t)dt.$$

$$+k_2 \int_0^{\left\{\begin{matrix} d(Az,Sz)d(Az,Tx_{2n+1})+ \\ d(Bx_{2n+1},Tx_{2n+1})d(Bx_{2n+1},Sz) \end{matrix}\right\}} \emptyset(t)dt$$

$$+ k_3 \int_0^{d(Az,Tx_{2n+1})d(Bx_{2n+1},Tx_{2n+1})} \emptyset(t)dt$$

$$+ k_4 \int_0^{d(Az,Sz)d(Az,Bx_{2n+1})} \emptyset(t)dt$$

Letting $n \to \infty$ on both sides and using (2.1.1),(2.1.2), then we get

$$\int_0^{[d(Az,z)]^2} \emptyset(t)dt \leq k_1 \int_0^{\{d(Az,z)d(z,z)+d(z,z)d(Az,z)\}} \emptyset(t)dt.$$

$$+k_2 \int_0^{\{d(Az,z)d(Az,z)+d(z,z)d(z,z)\}} \emptyset(t)dt$$

$$+ k_3 \int_0^{d(Az,z)d(z,z)} \emptyset(t)dt + k_4 \int_0^{d(Az,z)d(Az,z)} \emptyset(t)dt$$

$$\int_0^{[d(Az,z)]^2} \emptyset(t)dt \leq (k_2 + k_4) \int_0^{[d(Az,z)]^2} \emptyset(t)dt$$

We have Az = z, since $0 \leq k_1 + k_2 + k_3 + k_4 < 1$, where $k_1, k_2, k_3, k_4 \geq 0$

Since A(X) ⊆ T(X) implies there exists u ∈ X such that z = Az = Tu.

To prove Bu = z, put x = z, y = u in condition (2), we have

$$\int_0^{[d(Az,Bu)]^2} \emptyset(t)dt \leq k_1 \int_0^{\{d(Az,Sz)d(Bu,Tu)+d(Bu,Sz)d(Az,Tu)\}} \emptyset(t)dt.$$

$$+ k_2 \int_0^{\{d(Az,Sz)d(Az,Tu)+d(Bu,Tu)d(Bu,Sz)\}} \emptyset(t)dt$$

$$+ k_3 \int_0^{d(Az,Tu)d(Bu,Tu)} \emptyset(t)dt$$

$$+ k_4 \int_0^{d(Az,Sz)d(Az,Bu)} \emptyset(t)dt$$

Letting $n \to \infty$ on both sides and using (2.1.1),(2.1.2), and Az = z = Sz = Az = Tu , then we get

$$\int_0^{[d(z,Bu)]^2} \emptyset(t)dt \leq k_1 \int_0^{\{d(z,z)d(Bu,z)+d(Bu,z)d(z,z)\}} \emptyset(t)dt.$$

$+k_2 \int_0^{\{d(z,z)d(z,z)+d(Bu,z)d(Bu,z)\}} \emptyset(t)dt$

$+ k_3 \int_0^{d(z,z)d(Bu,z)} \emptyset(t)dt \; + k_4 \int_0^{d(z,z)d(z,Bu)} \emptyset(t)dt$

$\int_0^{[d(Bu,z)]^2} \emptyset(t)dt \leq k_2 \int_0^{[d(Bu,z)]^2} \emptyset(t)dt$

We have Bu = z, since $0 \leq k_1 + k_2 + k_3 + k_4 < 1$, where $k_1, k_2, k_3, k_4 \geq 0$

Hence Az = Sz = Bu = z.

Since (B,T) is weakly compatible BTu = TBu this implies Bz = Tz

Now we prove Bz = z

Put x = x_{2n} , y = z in condition (2) we have

$\int_0^{[d(Ax_{2n},Bz)]^2} \emptyset(t)dt \leq k_1 \int_0^{d(Ax_{2n},Sx_{2n})d(Bz,Tz)+d(Bz,Sx_{2n})d(Ax_{2n},Tz)} \emptyset(t)dt.$

$+k_2 \int_0^{d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tz)+d(Bz,Tz)d(Bz,Sx_{2n})} \emptyset(t)dt$

$+ k_3 \int_0^{d(Ax_{2n},Tz)d(Bz,Tz)} \emptyset(t)dt$

$+ k_4 \int_0^{d(Ax_{2n},Sx_{2n})d(Ax_{2n},Bz)} \emptyset(t)dt$

Letting $n \to \infty$ on both sides and using (2.1.1), Tz = Bz, then we get

$\int_0^{[d(z,Bz)]^2} \emptyset(t)dt \leq k_1 \int_0^{d(z,z)d(Bz,z)+d(Bz,z)d(z,z)} \emptyset(t)dt.$

$+k_2 \int_0^{d(z,z)d(z,z)+d(Bz,z)d(Bz,z)} \emptyset(t)dt$

$+ k_3 \int_0^{d(z,z)d(Bz,z)} \emptyset(t)dt + k_4 \int_0^{d(z,z)d(z,Bz)} \emptyset(t)dt$

$\int_0^{[d(z,Bz)]^2} \emptyset(t)dt \leq k_2 \int_0^{[d(z,Bz)]^2} \emptyset(t)dt$

We have Bz = z since $0 \leq k_1 + k_2 + k_3 + k_4 < 1$, where $k_1, k_2, k_3, k_4 \geq 0$

Hence Bz = Tz = z

Case 2:

Since A is weakly reciprocally continuous then $\lim_{n\to\infty} ASx_{2n} \to Az$

Since the pair (A,S) is A- compatible then $\lim_{n\to\infty} d(ASx_{2n}, SSx_{2n}) = 0$

giving that $\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} AAx_{2n} = Az$ (2.1.3)

Put x = Sx_{2n}, y = x_{2n+1} in condition (2), we have

$\int_0^{[d(ASx_{2n},Bx_{2n+1})]^2} \emptyset(t)dt \leq k_1 \int_0^{\left\{\begin{matrix} d(ASx_{2n},SSx_{2n})d(Bx_{2n+1},Tx_{2n+1})+ \\ d(Bx_{2n+1},SSx_{2n})d(ASx_{2n},Tx_{2n+1}) \end{matrix}\right\}} \emptyset(t)dt.$

$+k_2 \int_0^{\left\{\begin{matrix} d(ASx_{2n},SSx_{2n})d(ASx_{2n},Tx_{2n+1})+ \\ d(Bx_{2n+1},Tx_{2n+1})d(Bx_{2n+1},SSx_{2n}) \end{matrix}\right\}} \emptyset(t)dt$

$+ k_3 \int_0^{d(ASx_{2n},Tx_{2n+1})d(Bx_{2n+1},Tx_{2n+1})} \emptyset(t)dt$

$+ k_4 \int_0^{d(ASx_{2n},SSx_{2n})d(ASx_{2n},Bx_{2n+1})} \emptyset(t)dt$

Letting $n \to \infty$ on both sides and using (2.1.1),(2.1.3), then we get

$\int_0^{[d(Az,z)]^2} \emptyset(t)dt \leq k_1 \int_0^{\{d(Az,Az)d(z,z)+d(z,Az)d(Az,z)\}} \emptyset(t)dt.$

$+k_2 \int_0^{\{d(Az,Az)d(Az,z)+d(z,z)d(z,Az)\}} \emptyset(t)dt$

$+ k_3 \int_0^{d(Az,z)d(z,z)} \emptyset(t)dt + k_4 \int_0^{d(Az,Az)d(Az,z)} \emptyset(t)dt$

$\int_0^{[d(Az,z)]^2} \emptyset(t)dt \leq k_1 \int_0^{[d(Az,z)]^2} \emptyset(t)dt$

We have Az = z, since $0 \leq k_1 + k_2 + k_3 + k_4 < 1$, where $k_1, k_2, k_3, k_4 \geq 0$

Since A(X) ⊆T(X) implies there exists v ∈ X such that z = Az = Tv.

Put x = x_{2n} , y = v in condition(2), we have

$$\int_0^{[d(Ax_{2n},Bv)]^2} \emptyset(t)dt$$

$$\leq k_1 \int_0^{d(Ax_{2n},Sx_{2n})d(Bv,Tv)+d(Bv,Sx_{2n})d(Ax_{2n},Tv)} \emptyset(t)dt.$$

$$+ k_2 \int_0^{d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tv)+d(Bv,Tv)d(Bv,Sx_{2n})} \emptyset(t)dt$$

$$+ k_3 \int_0^{d(Ax_{2n},Tu)d(Bv,Tv)} \emptyset(t)dt + k_4 \int_0^{d(Ax_{2n},Sx_{2n})d(Ax_{2n},Bv)} \emptyset(t)dt$$

Letting $n \to \infty$ on both sides and using (2.1.1) and Tv = Az = z, then we get

$$\int_0^{[d(z,Bv)]^2} \emptyset(t)dt$$

$$\leq k_1 \int_0^{d(z,z)d(Bv,z)+d(Bv,z)d(z,z)} \emptyset(t)dt.$$

$$+k_2 \int_0^{d(z,z)d(z,z)+d(Bv,z)d(Bv,z)} \emptyset(t)dt$$

$$+ k_3 \int_0^{d(z,z)d(Bv,z)} \emptyset(t)dt + k_4 \int_0^{d(z,z)d(z,Bv)} \emptyset(t)dt$$

$\int_0^{[d(z,Bv)]^2} \emptyset(t)dt \leq k_2 \int_0^{[d(z,Bv)]^2} \emptyset(t)dt$

We have Bv = z, since $0 \leq k_1 + k_2 + k_3 + k_4 < 1$, where $k_1, k_2, k_3, k_4 \geq 0$

Since (B,T) is weakly compatible implies BTv = TBv , Then Bz = Tz

Now we prove Bz = z

Put x = x_{2n} , y = z in condition(2), we have

$$\int_0^{[d(Ax_{2n},Bz)]^2} \emptyset(t)dt$$

$$\leq k_1 \int_0^{d(Ax_{2n},Sx_{2n})d(Bz,Tz)+d(Bz,Sx_{2n})d(Ax_{2n},Tz)} \emptyset(t)dt.$$

$$+k_2 \int_0^{d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tz)+d(Bz,Tz)d(Bz,Sx_{2n})} \emptyset(t)dt$$

$$+ k_3 \int_0^{d(Ax_{2n},Tz)d(Bz,Tz)} \emptyset(t)dt + k_4 \int_0^{d(Ax_{2n},Sx_{2n})d(Ax_{2n},Bz)} \emptyset(t)dt$$

Letting $n \to \infty$ on both sides and using (2.1.1) and Tz = Bz , then we get

$$\int_0^{[d(z,Bz)]^2} \emptyset(t)dt \leq k_1 \int_0^{d(z,z)d(Bz,z)+d(Bz,z)d(z,z)} \emptyset(t)dt.$$

$$+k_2 \int_0^{d(z,z)d(z,z)+d(Bz,z)d(Bz,z)} \emptyset(t)dt + k_3 \int_0^{d(z,z)d(Bz,z)} \emptyset(t)dt + k_4 \int_0^{d(z,z)d(z,Bz)} \emptyset(t)dt$$

$\int_0^{[d(z,Bz)]^2} \emptyset(t)dt \leq k_2 \int_0^{[d(z,Bz)]^2} \emptyset(t)dt$

We have z = Bz , since $0 \leq k_1 + k_2 + k_3 + k_4 < 1$, where $k_1, k_2, k_3, k_4 \geq 0$

Hence Bz = Tz = z

Since B(X) ⊆ S(X) implies there exists w ∈ X such that Bz = z = Sw. Since the pair (A,S) is A compatible then $\lim_{n\to\infty} d(ASx_{2n}, SSx_{2n}) = 0$ implies that d(ASw, SSw) = 0 implies ASw = SSw implies Az = Sz = z.

Since Az = Bz = Sz = Tz = z, we get z in a common fixed point of A, B, S and T. The uniqueness of the fixed point can be easily proved.

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