On the Cubic Diophantine Equation with four unknowns $x^2 - y^2 = z^3 - w^3$

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Abstract - The sequences of integral solutions to the cubic equation with four variables $x^2 - y^2 = z^3 - w^3$ ***are obtained. A few properties among the solutions are presented. Also, Knowing a solution of the given equation a general formula for obtaining sequence of solutions based on the given solution are presented.***

Keywords: Cubic equation with four unknowns, non-homogeneous cubic, cubic with four unknowns, integral solutions.

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Notations used:

- $t_{m,n}$ ***- Polygonal number of rank*** n ***with size*** m.
- $\mathrm{P}_{\mathrm{n}}^{\mathrm{m}}$ ***- Pyramidal number of rank*** n ***with size*** m.
- Pr_n ***- Pronic number of rank*** n
- J_{n} ***- Jacobsthal number of rank*** n.
- G_n ***- Gnomonic number of rank n.***
- $Ct_{m,n}$ ***- Centered polygonal number of rank n with size m.***
- TK_n ***- Thabit – ibn – Kurrah number of rank n.***

I. INTRODUCTION

The Diophantine equation offers an unlimited field for research due to their variety [1-4]. In particular, one may refer [5-13] for cubic equation with three unknowns. In [14,15] cubic equations with four unknowns are studied for its non-trivial integral solutions and in [16,17] cubic equations with five unknowns and one may refer [18,19] for cubic equation with six unknowns are analyzed for its distinct integer solutions.

This communication concerns with the problem of obtaining infinitely many non-zero distinct integral solutions of cubic equation with four variables given by $x^2 - y^2 = z^3 - w^3$. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The cubic Diophantine equation with four unknowns to be solved for getting non-zero integral solution is

$$x^2 - y^2 = z^3 - w^3 \qquad (1)$$

We illustrate below different type of solutions for the given equation.

Illustration 1:

Equation (1) can be factorized as

$$(x+y)(x-y) = (z-w)(z^2 + zw + w^2)$$

which is written as system of two equations

$$x + y = z^2 + zw + w^2$$
$$x - y = z - w$$

From the above system, we get

$$x = \frac{1}{2}(z^2 + zw + w^2 + z - w)$$
$$y = \frac{1}{2}(z^2 + zw + w^2 - z + w)$$

As our aim is to find integer solutions, choose z and w as in the following cases:

Case (1): $z = 2a$, $w = 2b$

For this case, the corresponding non-zero distinct integer solutions to (1) are

$$x(a,b) = 2a^2 + 2ab + 2b^2 + a - b$$
$$y(a,b) = 2a^2 + 2ab + 2b^2 - a + b$$
$$z(a) = 2a$$
$$w(b) = 2b$$

Properties:

- $x(a,1) + y(a,1) - Ct_{8,a} - J_3 = 0.$
- $x(a,1) + z(a,1) - 2\,\mathrm{Pr}_a - J_1 \equiv 0 \pmod{3}.$
- $6(x+y)$ is a nasty number.
- $2x(a,1) + 3y(a,1) - 10t_{4,a} \equiv 2 \pmod{9}.$
- $\left[\{z(a)+w(a)\}^2 - \{x(a,a)+y(a,a)\}\right]$ is a perfect square.
- $y(1,2^a) + w(2^a) - 2t_{4,2^a} - TK_a - G_{2^a} - J_3 = 0.$

Case (2): $z = 2a+1$, $w = 2b$

For the above values of z and w, the corresponding non-zero distinct integer solutions to (1) are

$$x(a,b) = 2a^2 + 2ab + 2b^2 + 3a + 1$$
$$y(a,b) = 2a^2 + 2ab + 2b^2 + a + 2b$$
$$z(a) = 2a + 1$$
$$w(b) = 2b$$

Properties:

- $x(a,1) + y(a,1) - 4\,\mathrm{Pr}_a \equiv 7 \pmod{4}.$
- $2x(a,1) - [z(a)]^2 - 3w(a) = 5.$
- $y(a,a+1) + w(a+1) - 2\{t_{4,a} + t_{4,a+1} + \mathrm{Pr}_a\} \equiv 4 \pmod{5}.$
- $3\{z(a^2) + w(a) - G_a - 2\}$ is a nasty number.
- Each of the following expressions represents a perfect square:

$$8\{x(1,b) - y(1,b) + G_b\}$$
$$x(a^2,a) - y(a^2,a) + z(a^2) + G_a + J_3$$

Case (3): $z = 2a$, $w = 2b+1$

For the above values of z and w, the corresponding non-zero distinct integer solutions to (1) are

$$x(a,b) = 2a^2 + 2ab + 2b^2 + 2a + b$$
$$y(a,b) = 2a^2 + 2ab + 2b^2 + 3b + 1$$
$$z(a) = 2a$$
$$w(b) = 2b + 1$$

Illustration 2:

On substituting the linear transformations

$$z = y + h \,,\; w = y + k \,,\; h \neq k \neq 0 \qquad (2)$$

in (1), it leads to

$$x^2 = (3h - 3k + 1)y^2 + (3h^2 - 3k^2)y + h^3 - k^3 \,,\quad h \neq k \qquad (3)$$

To Solve (3), we have to go in for a particular value for h and k. For simplicity and brevity, by taking $h = 3$ and $k = 1$ in (3), we obtain

$$x^2 = 7y^2 + 24y + 26 \qquad (4)$$

Performing some algebraic simplifications, the above equation is written as

$$Y^2 = 7x^2 - 38 \qquad (5)$$

where $Y = 7y + 12$ (6)

The smallest positive integer solution of (5) is

$$Y_0 = 5, x_0 = 3 \qquad (7)$$

To obtain the other solutions of (5), consider the pell equation

$$Y^2 = 7x^2 + 1 \qquad (8)$$

whose general solution is given by

$$\tilde{x}_n = \frac{g_n}{2\sqrt{7}}$$
$$\tilde{Y}_n = \frac{f_n}{2}$$

where

$$f_n = (8 + 3\sqrt{7})^{n+1} + (8 - 3\sqrt{7})^{n+1}$$
$$g_n = (8 + 3\sqrt{7})^{n+1} - (8 - 3\sqrt{7})^{n+1} \,, n = -1,0,1,2,...$$

Applying Brahmagupta lemma between (x_0, Y_0) and $(\tilde{x}_n, \tilde{Y}_n)$, the other integer solutions of (5) are given by

$$x_{n+1} = \frac{3}{2} f_n + \frac{5}{2\sqrt{7}} g_n$$
$$Y_{n+1} = \frac{5}{2} f_n + \frac{21}{2\sqrt{7}} g_n \qquad (9)$$

By using (9) in (6) and employing (2), we obtain the non-zero distinct integral solutions to (1) given by

$$x_{n+1}=\frac{3}{2}f_n+\frac{5}{2\sqrt{7}}g_n$$

$$y_{n+1}=\frac{1}{7}\left\{\frac{5}{2}f_n+\frac{21}{2\sqrt{7}}g_n-12\right\}$$

$$z_{n+1}=\frac{1}{7}\left\{\frac{5}{2}f_n+\frac{21}{2\sqrt{7}}g_n+9\right\}$$

$$w_{n+1}=\frac{1}{7}\left\{\frac{5}{2}f_n+\frac{21}{2\sqrt{7}}g_n-5\right\},\ n=-1,0,1,2,...$$

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he recurrence relations satisfied by x, y, z and w are given by

$$x_{n+3}-16x_{n+2}+x_{n+1}=0.$$

$$y_{n+3}-16y_{n+2}+y_{n+1}=24$$

$$z_{n+3}-16z_{n+2}+z_{n+1}=-18$$

$$w_{n+3}-16w_{n+2}+w_{n+1}=10,\ n=-1,0,1,2,...$$

A few numerical examples are given below in Table1:

Table 1 : Numerical Examples

n	x_{n+1}	y_{n+1}	z_{n+1}	w_{n+1}
-1	3	-1	2	0
0	39	13	16	14
1	621	233	236	234
2	9897	3739	3742	3740
3	157731	59615	59618	59616
4	2513799	950125	950128	950126

From the above table , we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} and y_{n+1} are always odd
2. z_{n+1} and w_{n+1} are always even
3. Relations among the solutions:

- ❖ $y_{n+3}=16y_{n+2}-y_{n+1}+24$
- ❖ $798y_{n+1}=38x_{n+2}-304x_{n+1}-1368$
- ❖ $16891y_{n+2}=399x_{n+3}+1064y_{n+1}-27132$
- ❖ $798y_{n+2}=304x_{n+2}-38x_{n+1}-1368$
- ❖ $798y_{n+3}=4826x_{n+2}-304x_{n+1}-1368$
- ❖ $x_{n+3}=16x_{n+2}-x_{n+1}$
- ❖ $798y_{n+2}=38x_{n+3}-304x_{n+2}-1368$
- ❖ $798y_{n+3}=304x_{n+3}-38x_{n+2}-1368$
- ❖ $16891y_{n+1}=6384x_{n+3}+133y_{n+1}-28728$
- ❖ $798y_{n+1}=304x_{n+3}-4826x_{n+2}-1368$

❖ Each of the following expressions represents a nasty number:

- ➢ $\frac{6}{19}\{7y_{2n+3}-91y_{2n+2}-182\}$
- ➢ $\frac{2}{19}\{103x_{2n+2}-5x_{2n+3}+114\}$
- ➢ $\frac{6}{2413}\{21x_{2n+4}-11501y_{2n+2}-14890\}$
- ➢ $\frac{2}{19}\{1643x_{2n+3}-103x_{2n+4}+114\}$

❖ Each of the following expressions in Table 2 represents a hyperbola:

Table 2: Hyperbola

Hyperbola	(p_n, q_n)
$p_n^2-63q_n^2=12996$	$(103x_{n+1}-5x_{n+2},13x_{n+1}-x_{n+2})$
$7p_n^2-q_n^2=90972$	$(1643x_{n+2}-103x_{n+3},\ 4347x_{n+2}-273x_{n+3})$
$p_n^2-7q_n^2=23290276$	$(21x_{n+3}-11501y_{n+1}-19716,5x_{n+3}-4347y_{n+1}-7452)$
$63p_n^2-q_n^2=90972$	$(91y_{n+1}-7y_{n+2}+144,721y_{n+1}-35y_{n+2}+1176)$

❖ Each of the following expressions in Table 3 represents a parabola:

Table 3: Parabola

Parabola	(p_n, q_n)
$2793q_n^2 = 4527p_n - 1032156$	$(103x_{2n+2} - 5x_{2n+3} + 114, 13x_{n+1} - x_{n+2})$
$q_n^2 = 399p_n - 90972$	$(1643x_{2n+3} - 103x_{2n+4} + 114,\ 4347x_{n+2} - 273x_{n+3})$
$7q_n^2 = 2413p_n - 23290276$	$(21x_{2n+4} - 11501y_{2n+2} - 14890, 5x_{n+3} - 4347y_{n+1} - 7452)$
$q_n^2 = -1197p_n - 90972$	$(91y_{2n+2} - 7y_{2n+3} + 182, 721y_{n+1} - 35y_{n+2} + 1176)$

Illustration 3:

By substituting the transformations

$$x = 2n^3u + v, y = 2n^3u - v, \quad z = 2nP, w = 2nQ \tag{10}$$

in (1), we get

$$uv = P^3 - Q^3,\ P \neq Q \neq 0 \tag{11}$$

from which note that

$$u = P^2 + PQ + Q^2, \quad v = P - Q \tag{12}$$

By using (12) in (10) , we have the corresponding non-zero distinct integer solutions to (1) found to be

$$x = 2n^3(P^2 + PQ + Q^2) + (P - Q)$$
$$y = 2n^3(P^2 + PQ + Q^2) - (P - Q)$$
$$z = 2nP$$
$$w = 2nQ$$

Note :
Using the transformations

$$x = u + 2n^3v, y = u - 2n^3v, z = 2nP, w = 2nQ$$

in (1), we get the corresponding non-zero distinct integer solutions to (1) given by

$$x = (P^2 + PQ + Q^2) + 2n^3(P - Q)$$
$$y = (P^2 + PQ + Q^2) - 2n^3(P - Q)$$
$$z = 2nP$$
$$w = 2nQ$$

Illustration 4:

Introducing the transformations

$$z = 2k - 1 + K\ ,\ w = K$$

in (1), it gives

$$x^2 - y^2 = 8k^3 - 12k^2 + 6k - 1 + 6K^2k - 3K^2 + 3K + 12Kk^2 - 12Kk$$

Observe that for all values of k and K, the Right Hand Side of the above equation is always odd.

Employing the identity, $(A+1)^2 - A^2 = 2A + 1$ we have

$$x = 4k^3 - 6k^2 + 3k + 3K^2k + 6Kk^2 - 6Kk - \frac{3K(K-1)}{2}$$

$$y = 4k^3 - 6k^2 + 3k - 1 + 3K^2k + 6Kk^2 - 6Kk - \frac{3K(K-1)}{2}$$

A few numerical examples are presented below:

Table 3 : Numerical Examples

K	k	x	y	z	w
1	2	32	31	4	1
2	3	168	167	7	2
4	2	140	139	7	4
5	3	438	437	10	5

III. GENERATION OF SOLUTIONS

Let (x_0, y_0, z_0, w_0) be the initial solution to (1).
Assume

$$x_1 = 2h - 3^3x_0, y_1 = 3^3y_0 + h, z_1 = 3^2z_0, w_1 = 3^2w_0$$

to be the second solution to (1).
Then, we obtain

$$h = 36x_0 + 18y_0$$

Therefore,
$$x_1 = 45x_0 + 36y_0$$
$$y_1 = 36x_0 + 45y_0$$

Repeating the above process, we get the corresponding non-zero distinct integral solutions to (1) represented in the matrix form as given below:

$$\begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \end{pmatrix} = 9^n \begin{pmatrix} \frac{9^n+1}{2} & \frac{9^n-1}{2} & 0 & 0 \\ \frac{9^n-1}{2} & \frac{9^n+1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{pmatrix}$$

IV. CONCLUSION

In this paper, we have presented sets of infinitely many non-zero distinct integer solutions to the cubic equation with four unknowns given by $x^2 - y^2 = z^3 - w^3$. In other words, we have obtained quadruples such that, in each quadruple, the difference of the squares of any two members equals the difference of cubes of its other two members. As Diophantine equations are rich in variety due to their definition, one may attempt to find integer solutions to higher degree Diophantine equations with multiple variables along with their suitable properties..

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