Fixed Point Results on Complete Random G-Metric Space for Contractive Mappings

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Abstract - In the present paper some fixed point theorems are established in random G-Metric space. The results are obtained from the basic definitions of G- metric space.

***Keywords: Fixed point, Random G metric space Ams subject classicification: 47 H10*.**

I. Introduction and Prilimnaries

Many different generalizations can be done as past years as a development of invariant point theory in Metric space. The researchers Gahler and Dhage describes the notion of metric space and generalized many results related to the spaces 2-metric and D-metric.The Metric space conception is generalized in different ways has been proposed by many authors by Gahler and Dhage they obtain the results for 2-Metric space are independent and also proves the generalization and corresponding many results in metric spaces.The author Dhage develops the theory of D-Metric space and generalized many results for single and multivalued mappings of ordinary metric functions. Authors Abbas and Rhoades introduced the concept of communal fixed point theory in general metric space with distinct mappings gratifying the property P in G-Metric space. Also the author Mustafa.et. al. presents some theorems offixed point for mapping with different contractive condition.

In recent years, the study of fixed points of random operators forms a central topic in probabilistic functional analysis. In 1950 study of probabilistic was initiated in The Prague School of probabilistic. After the publication of the survey articles of Bharucha-Reid so many research works have been started in this area. Then many interesting random fixed point results and applications have appeared in the literature- for example the work of Beg and Shahazad, Lin (particularly random iteration schemes leading to random fixed point of random operators have been elaborately discussed in B.S. Choudhary, and M. Ray, B.S. Choudhary, and A. Upadhyay, V.B. Dhagat , A.Sharma, & R.K Bhardwaj and others.

In the present paper using above concepts a new concept of random G metric spaces is stablished, which is the modification of G metric space using random operator. Now the modifications of the definitions are as follows:

Definition 2.1: Suppose U be a non-empty set, let a mapping$G : U \times U \times U \to R^+$ be a functional which satisfy the subsequent axioms:

1] G_1 $G(\xi u, \xi v, \xi w) = 0$ $If \xi u = \xi v = \xi w.$

2]
G_2 $0 < G(\xi u, \xi v, \xi w)$ $\forall \xi u, \xi v \in U$ $with$ $\xi u \neq \xi v.$

3]
G_3 $G(\xi u, \xi v, \xi w) \leq$
$G(\xi u, \xi v, \xi w),$ $\forall$ $\xi u, \xi v, \xi w \in U$ $with \xi w \neq \xi v$

4]
G_4 $G(\xi u, \xi v,\ \xi w) = G(\xi u, \xi w, \xi\, v) = G(\xi v,\ \xi w,\ \xi u)$

(Symmetry in all three variables)

5]
G_5 $G(\xi u, \xi v,\ \xi w) \leq G(\xi u, \xi x,\ \xi x) + G(\xi x, \xi v,\ \xi w),\ \ \forall\ \xi u, \xi v, \xi w \in U$

(Rectangular inequality)

Therefore the functional G is known as random G-Metric, with the couple (U, G).

Definition 2.2: Suppose a metric space (U, G) be random G-metric then a classification$\{\xi x_n\}$ is named as random G- Cauchy, for every $\in > 0, \exists\, n \in N$ such that $G(\xi\, x_n, \xi x_m, \xi x_l) < \in,\ \ \forall\, n,\ m, l \in N$

i.e. $G(\xi x_n, \xi\, x_m, \xi x_l) \rightarrow 0\ \ \ as\ \ n, m, l\ \rightarrow \infty$

Proposition 2.3: Suppose a metric space (U, G) be G-metric, also a sequence$\{x_n\}$ be a point of U, and then the sequence$\{x_n\}$ is G- Convergent to x, If

$\lim_{n,m\rightarrow\infty} G(x, x_n, x_m) = 0\ \ \ \ \forall \in > 0\ \ \exists\, n \in N$ Such that

$G(x, x_n, x_m) < \in \qquad\qquad \forall\, n, m\ \geq M$

Proposition 2.4: Suppose a metric space (U, G) be random G-metric, therefore the subsequent results are Comparable as

(i) Classification $\{\xi x_n\}$ is G- Converges to ξx.

(ii) $G(\xi x_n, \xi x_n, \xi x) \rightarrow 0\ as\ \ n \rightarrow \infty$

(iii) $G(\xi x_n, \xi x, \xi x) \rightarrow 0\ as\ \ n \rightarrow \infty$

Proposition 2.5: Suppose a metric space (U, G) and (U',G') be two Random G-metrics then S is a aligning from X to X' is at a point which is G- unceasing, and also it is unceasing at $\xi\, x$ If a metric space is raomdom G- Sequentially, wherever $\{\xi x_n\}$ is G-convergent to ξx, $\{f(\xi x_n)\}$is G- converges to $f(\xi x)$.

Definition2.6: Suppose the spaces (U, G) and (U', G') be double random G-metric and let S :(U, G) → (U',G') be a functional , then the function f is known to be random G- unceasing at a point $\xi a \in U$, Uncertainty assumed $\in > 0, \exists\, \delta > 0$ such that $\xi x, \xi y \in U\ ;\ \ G(\xi a, \xi x, \xi y) < \delta$

This implies that $G\big(f(\xi a), f(\xi x), f(\xi y)\big) < \in$, A functional f is G- unceasing on X iff it is G- unceasing at all $\xi a \in U$.

Definition 2.7: Suppose a metric space (U, G) be random G-metric is known to be random G- complete, If for all random G-Cauchy classification in (U, G) is random G- converges in (U, G).

Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subset of Ω . U stands for a complete metric space, C is nonempty subset of U

Definition 2.8: A function R : $\Omega \times C\ \rightarrow C$ is said to be measurable if

$R^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of U

Definition 2.9: A function R : $\Omega \times C \rightarrow C$ is said to be random operator, if

R(., x) : $\Omega \rightarrow C$ is measurable for every $x \in C$.

Definition 2.10: A random operator R : $\Omega \times C \rightarrow C$ is said to be continuous if for fixed $\xi \in \Omega, R(\xi,.) : C \rightarrow C$ is continuous.

Definition 2.11 :. A measurable function $g : \Omega \rightarrow C$ is said to be random fixed point of the random operator R : $\Omega \times C \rightarrow C$, if $R(\xi, g(\xi)) = g(\xi)$, □ξ ∈ Ω□ □or R(ξx) = ξx

3. Proposition: Suppose a metric space (U, G) be random G-metric, therefore the functional $G(\xi x, \xi y, \xi z)$ is jointly continuous in all three of its variable.

4. **Main Results**

Theorem 4.1.: Suppose a metric (U, G) be Complete random G-Metric then for a mapping S : U → U which satisfy subsequent condition for all $\xi x, \xi y, \xi z \in U$.

$$G(S(\xi x), S(\xi y), S(\xi z)) \leq q\, max \left\{ \begin{array}{c} G(\xi x, \xi y, \xi z), G(\xi x, S(\xi x), S(\xi y)), G(\xi y, S(\xi y), S(\xi z)), \\ G(\xi z, S(\xi z), S(\xi x)), G(\xi x, S(\xi y), S(\xi z)), G(\xi y, S(\xi x), S(\xi z)) \\ G(\xi z, S(\xi x), S(\xi y)), G(\xi x, S(\xi x), S(\xi x)), G(\xi y, S(\xi y), S(\xi y)), \\ G(\xi z, S(\xi z), S(\xi z)), G(\xi x, S(\xi y), S(\xi y)), G(\xi y, S(\xi z), S(\xi z)), \\ G(\xi z, S(\xi x), S(\xi x)), G(\xi x, \xi y, S(\xi z)), \\ G(\xi x, \xi z, S(\xi y)), G(\xi z, \xi y, S(\xi x)) \end{array} \right\}$$

[4.1.1]

Where $q \in \left(0, {}^{1}/_{2}\right]$ then prove that S has a unique fixed point and S is random G- continuous at ξu.

Proof: let$\xi\, x_0$ be any random point in U, then we express a sequence $\{\xi x_n\}$ in U as

$\xi x_n = S^n(\xi x_0)$

Then from [4.1.1], we have

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq q\, max \left\{ \begin{array}{c} G(\xi x_{n-1}, \xi x_n, \xi x_n), G(\xi x_{n-1}, \xi x_n, \xi x_{n+1}), G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_n, \xi x_{n+1}, \xi x_n), G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1}), G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \\ G(\xi x_n, \xi x_n, \xi x_{n+1}), G(\xi x_{n-1}, \xi x_n, \xi x_n), G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}), G(\xi x_{n-1}, \xi\, x_{n+1}, \xi x_{n+1}), G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_n, \xi x_n, \xi x_n), G(\xi x_{n-1}, \xi\, x_n, \xi x_{n+1}), G(\xi x_{n-1}, \xi x_n, \xi x_{n+1}), \\ G(\xi x_n, \xi x_n, \xi x_{n-1}) \end{array} \right\}$$

$$\leq q\, max \left\{ \begin{array}{c} G(\xi x_{n-1}, \xi x_n, \xi x_n), G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}), G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_{n-1}, \xi x_n, \xi x_{n+1}), G(\xi x_{n+1}, \xi x_n, \xi x_n) \end{array} \right\}$$

[4.1.2]

So,

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq q \; max \begin{Bmatrix} G(\xi x_{n-1}, \xi x_n, \xi x_n), G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_{n+1}, \xi x_n, \xi x_n), G(\xi x_{n-1}, \xi x_n, \xi x_{n+1}) \end{Bmatrix}$$

[4.1.3]

Since, By G(5) we have

$$G(\xi x_{n-1}, \xi x_n, \xi x_n) \leq q \; [G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_{n-1}, \xi x_{n+1})]$$

[4.1.4]

Also,

$$G(\xi x_{n-1}, \xi x_n, \xi x_{n+1}) \leq q \; [G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_n, \xi x_{n+1})]$$

That implies

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq q \; max \begin{Bmatrix} G(\xi x_{n-1}, \xi x_n, \xi x_n), G(\xi x_n, \xi x_n, \xi x_{n+1}) + G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_{n+1}, \xi x_n, \xi x_n), G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_n, \xi x_{n+1}) \end{Bmatrix}$$

$$\leq q \; max \begin{Bmatrix} G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_n, \xi x_{n+1}) \end{Bmatrix}$$

By G(3), $G(\xi x_n, \xi x_n, \xi x_{n+1}) \leq G(\xi x_n, \xi x_{n+1}, \xi x_{n+1})$

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq q max \begin{Bmatrix} G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \end{Bmatrix}$$

[4.1.5]

Therefore

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq q \; [G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_{n+1}, \xi x_{n+1})]$$

[4.1.6]

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq \frac{q}{1-q} \; G(\xi x_{n-1}, \xi x_n, \xi x_n)$$

[4.1.7]

Let $k = \frac{q}{1-q}$, then for $k < 1$ and by repeated process of the application of Eq. [8.1.3.7], we have

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq k^n \; G(\xi x_0, \xi x_1, \xi x_1)$$

[4.1.8]

Then for all $n, m \in N, n \leq m,$ we have by again use of rectangular inequality and Eq.[8.1.3.8]

We have

$$G(\xi x_n, \xi x_m, \xi x_m) \leq G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) + G(\xi x_{n+1}, \xi x_{n+2}, \xi x_{n+2}) + G(\xi x_{n+2}, \xi x_{n+3}, \xi x_{n+3}) + \dots\dots\dots\dots\dots\dots\dots\dots\dots + G(\xi x_{m-1}, \xi x_m, \xi x_m)$$

$$\leq (k^n + k^{n+1} + \cdots \dots\dots\dots\dots + k^{m-1}) \ G(\xi x_0, \xi x_1, \xi x_1)$$

[4.1.9]

$$\leq \frac{k^n}{1-k} \ G(\xi x_0, \xi x_1, \xi x_1)$$

Then, $\lim_{n,m\to\infty} G(\xi x_n, \xi x_m, \xi x_m) = 0$

Since $\lim_{n\to\infty} \leq \frac{k^n}{1-k} \ G(\xi x_0, \xi x_1, \xi x_1) = 0$ for all $n, m \in N$

Therefore by prop (G 5) we assume that

$G(\xi x_n, \xi x_m, \xi x_l) \leq G(\xi x_n, \xi x_m, \xi x_m) + G(\xi x_l, \xi x_m, \xi x_m)$

Now at limit $n, m, l \to \infty$ we get

$G(\xi x_n, \xi x_m, \xi x_l) \to 0$

So, the sequence $\{\xi x_n\}$ is a G- Cauchy classification, by extensiveness of G- metric space (U, G), there exist $\xi v \in U$ then the classification $\{\xi x_n\}$ is G-converges toξ v.

Let $S(\xi v) \neq \xi v$, then

$$G(\xi x_n, S(\xi v), S(\xi v)) \leq q\ max \left\{ \begin{array}{c} G(\xi x_{n-1}, \xi v, \xi v), G\big(\xi x_{n-1}, \xi x_n, S(\xi v)\big), G(\xi v, S(\xi v), S(\xi v)\ ), \\ G(\xi v, S(\xi v), \xi x_n), G(\xi x_{n-1}, S(\xi v), S(\xi v)), G(\xi v, \xi x_n, S(\xi v)) \\ G(\xi v, \xi x_n, S(\xi v)), G(\xi x_{n-1}, \xi x_n, \xi x_n), G(\xi v, S(\xi v), S(\xi v)), \\ G(\xi v, S(\xi v), S(\xi v)), G(\xi x_{n-1}, S(\xi v), S(\xi v)), G(\xi v, S(\xi v), S(\xi v)), \\ G(v, x_n, x_n), G(x_{n-1}, v, S(v)), G(x_{n-1}, v, S(v)), \\ G(\xi v, \xi v, \xi x_n) \end{array} \right\}$$

$$\leq q\ max \left\{ \begin{array}{c} G(\xi x_{n-1}, \xi v, \xi v), G\big(\xi x_{n-1}, \xi x_n, S(\xi v)\big), G\big(\xi v, \xi x_n, S(\xi v)\big), \\ G\big(\xi v, S(\xi v), S(\xi v)\big), G\big(\xi x_{n-1}, S(\xi v), S(\xi v)\big), G(\xi x_{n-1}, \xi x_n, \xi x_n), \\ G(\xi v, \xi x_n, \xi x_n), G(\xi x_{n-1}, \xi v, S(\xi v)), G(\xi v, \xi v, \xi x_n) \end{array} \right\}$$

$$\leq q\ max \left\{ \begin{array}{c} G(\xi x_{n-1}, \xi v, \xi v), G(\xi x_{n-1}, \xi x_n, \xi x_n), \\ G\big(\xi v, S(\xi v), S(\xi v)\big), G\big(\xi x_{n-1}, S(\xi v), S(\xi v)\big), \\ G(\xi v, \xi x_n, \xi x_n) \end{array} \right\}$$

[4.1.10]

Hence by taking the limit $n \to \infty$ we have the function G is continuous on its variable

Therefore $G(\xi v, S(\xi v), S(\xi v)) \leq q\ G(\xi v, S(\xi v), S(\xi v))$

This gives contradiction. Since $0 \leq q \leq 1/2$,

Therefore $\xi v = S(\xi v)$

Uniqueness: Let us assume that $\xi w \neq \xi v$ is such that

$T(\xi w) = \xi w$, then from [4.1.1] we have

$G(\xi v, \xi w, \xi w) \leq q\ max\{G(\xi v, \xi w, \xi w), G(\xi w, \xi v, \xi v)\}$

Then

$G(\xi v, \xi w, \xi w) \leq k\ G(\xi w, \xi v, \xi v)$

Now again by same procedure we will find

$G(\xi w, \xi v, \xi v) \leq qG(\xi v, \xi w, \xi w)$, thus

$$G(\xi w, \xi v, \xi v) \leq q^2 G(\xi v, \xi w, \xi w) \tag{4.1.11}$$

Which proves that $\xi w = \xi v$, since $0 \leq q \leq 1/2$ S is continuous at ξv.

Let $\{\xi y_n\}$ be any classification of U, then

$\lim\{\xi y_n\} = \xi v$, then

$$G(S(\xi y_n), S(\xi v), S(\xi y_n)) \leq q\ max \left\{ \begin{matrix} G(\xi y_n, \xi v, \xi y_n), G(\xi y_n, S(\xi y_n), S(\xi y_n)), \\ G(\xi v, S(\xi v), S(\xi v)), G(\xi y_n, S(\xi v), S(\xi v)), \\ G(\xi v, S(\xi y_n), S(\xi y_n)) \end{matrix} \right\} \tag{4.1.12}$$

And we have

$$G(S(\xi y_n), \xi v, S(\xi y_n)) \leq q\ max \left\{ \begin{matrix} G(\xi y_n, \xi v, \xi y_n), G(\xi y_n, S(\xi y_n), S(\xi y_n)), \\ G(\xi y_n, \xi v, \xi v) \end{matrix} \right\} \tag{4.1.13}$$

Now by prop (G5) we have

$$G(\xi y_n, S(\xi y_n), S(\xi y_n)) \leq G(\xi y_n, \xi v, \xi v) + G(\xi v, S(\xi y_n), S(\xi y_n)) \tag{4.1.14}$$

And eq.[4.1.13] give the subsequent cases

(i) $G(S(\xi y_n), \xi v, S(\xi y_n)) \leq q\ G(\xi y_n, \xi y_n, \xi v)$

(ii) $G(S(\xi y_n), \xi v, S(\xi y_n)) \leq q\ G(\xi y_n, \xi v, \xi v)$

(iii) $G(S(\xi y_n), \xi v, S(\xi y_n)) \leq qG(\xi y_n, \xi v, \xi v)$

In every case we get as limit $n \to \infty$ we have

$G\big(\xi v, S(\xi y_n), S(\xi y_n)\big) \to 0$

Hence by proposition [3] we have classification $\{S(\xi y_n)\}$ is G- convergent to ξv,

i.e. $\xi v = S(\xi v)$

and by proposition we have S is G- continuous at ξv.

Corollary 4.2: Suppose a metric (U, G) space be random G- Complete or it is Complete random G-Metric Space for a mapping S: U → U which satisfies the subsequent axioms for some $m \in N$ and for all $\xi x, \xi y, \xi z \in U$.

$$G\big(S^m(\xi x), S^m(\xi y), S^m(\xi z)\big) \leq q\ max \left\{ \begin{array}{c} G(\xi x, \xi y, \xi z), G\big(\xi x, S^m(\xi x), S^m(\xi y)\big), G\big(\xi y, S^m(\xi y), S^m(\xi z)\big), \\ G\big(\xi z, S^m(\xi z), S^m(\xi x)\big), G\big(\xi x, S^m(\xi y), S^m(\xi z)\big), G\big(\xi y, S^m(\xi x), S^m(\xi z)\big) \\ G\big(z, S^m(\xi x), S^m(\xi y)\big), G\big(\xi x, S^m(\xi x), S^m(\xi x)\big), G\big(\xi y, S^m(\xi y), S^m(\xi y)\big), \\ G\big(\xi z, S^m(\xi z), S^m(\xi z)\big), G\big(\xi x, S^m(\xi y), S^m(\xi y)\big), G\big(\xi y, S^m(\xi z), S^m(\xi z)\big), \\ G\big(\xi z, S^m(\xi x), S^m(\xi x)\big), G\big(\xi x, \xi y, S^m(\xi z)\big), \\ G(\xi x, \xi z, S^m(\xi y)), G(\xi z, \xi y, S^m(\xi x)) \end{array} \right\}$$

[4,2.1]

Where $q \in \left(0, {}^{1}\!/_{2}\right]$ then ξv is a unique fixed point of mapping S and S^m is random G- continous at ξv.

Proof: By the hypothesis we have to verify that S^m has a unique fixed point (say ξv)

that is, $S^m(\xi v) = \xi v$,

but $S(\xi v) = S(S^m(\xi v))$

$= S^{m+1}(\xi v) = S^m(S(\xi v))$

So, is another fixed point of S^m and by uniqueness it proves that $S(\xi v) = \xi\ v$.

Theorem 4.3: Suppose a metric (U, G) be Complete random G-Metric then for a mapping

S : U → U which setisfies the subsequent axioms for all $\xi x, \xi y, \xi z \in U$, then

$$G\big(S(\xi x), S(\xi y), S(\xi z)\big) \leq q\ max \left\{ \begin{array}{c} G\big(\xi x, S(\xi x), S(\xi x)\big) + G\big(\xi y, S(\xi y), S(\xi x)\big), \\ G\big(\xi y, S(\xi y), S(\xi z)\big) + G\big(\xi z, S(\xi z), S(\xi y)\big), \\ G\big(\xi z, S(\xi z), S(\xi x)\big) + G\big(\xi x, S(\xi x), S(\xi z)\big) \end{array} \right\}$$

[4.3.1]

Where $q \in \left(0, {}^{1}/_{2}\right]$ then v is a exclusive invariant point of mapping S and S is random G- continous at ξ v.

Proof: Suppose S be a mapping satisfies Eq. [4.3.1], let ξx_0 be any random point in U, then we express a classification $\{\xi x_n\}$ in U such that

$$\xi x_n = T^n(\xi x_0)$$

Then from [8.1.5.1],we have

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq q\ max \begin{Bmatrix} G(\ \xi x_{n-1}, \xi x_n, \xi x_{n+1}) + G(\xi x_n, \xi x_{n+1}, \xi x_n), \\ G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) + G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_n, \xi x_{n+1}, \xi x_n) + G(\xi x_{n-1}, \xi x_n, \xi x_{n+1}) \end{Bmatrix}$$

[4.3.2]

$$= q\ max \begin{Bmatrix} G(\xi x_{n-1}, \xi x_n, \xi x_{n+1}) + G(\xi x_n, \xi x_{n+1}, \xi x_n), \\ 2G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \end{Bmatrix}$$

Therefore by property G (5) we have

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq q\ max\{G(\xi x_{n-1}, \xi x_n, \xi x_{n+1}) + 2G(\xi x_n, \xi x_{n+1}, \xi x_{n+1})\}$$

Since $0 \leq q \leq {}^{1}/_{2}$, then it must be the case that

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq q\ G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1})$$

[4.3.3]

Also from (G 5) we have

$$G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1}) \leq G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_{n+1}, \xi x_{n+1})$$

[4.3.4]

So [8.1.5.3] implies that

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq \frac{q}{1-q}\ G(\xi x_{n-1}, \xi x_n, \xi x_n)$$

[4.3.5]

Let $k = \frac{q}{1-q}$ then $k < 1$ and by repeated application of same process we have

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq k^n\ G(\xi x_0, \xi x_1, \xi x_1)$$

[4.3.6]

Then, for all $n, m \in N,\ n < m,$ we have by repeated process of triangular inequality, we have

$$G(\xi x_n, \xi x_m, \xi x_m) \leq\ G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) + G(\xi x_{n+1}, \xi x_{n+2}, \xi x_{n+2}) + \\ G(\xi x_{n+2}, \xi x_{n+3}, \xi x_{n+3}) + \cdots \dots \dots \dots \dots + G(\xi x_{m-1}, \xi x_m, \xi x_m)$$

$$\leq \ (k^n + k^{n+1} + \cdots \ldots \ldots \ldots \ldots + k^{m-1}) \ \ G(\xi x_0, \xi x_1, \xi x_1)$$

[4.3.7]

$$\leq \ \frac{k^n}{1-k} \ \ G(\xi x_0, \xi x_1, \xi x_1)$$

Then, $\lim_{n,m\to\infty} G(\xi x_n, \xi x_m, \xi x_m) = 0$

So, the sequence $\{\xi x_n\}$ is a G- Cauchy sequence for mapping S, also by completeness of G- metric (U, G) space, for every $\xi v \in U$ we have the sequence $\{\xi x_n\}$ is G-convergent to ξv.

Let us suppose that $S(\xi v) \neq \xi v$, then

$$G(\xi x_n, S(\xi v), S(\xi v)) \leq q \ max \begin{Bmatrix} G\big(\xi x_{n-1}, \xi x_n, S(\xi v)\big) + G(\xi v, S(\xi v), \xi x_n), \\ G\big(\xi v, S(\xi v), S(\xi v)\big) + G\big(\xi v, S(\xi v), S(\xi v)\big), \\ G(\xi v, S(\xi v), \xi x_n) + G\big(\xi x_{n-1}, \xi x_n, S(\xi v)\big) \end{Bmatrix}$$

[4.3.8]

Now for the limit as $n \to \infty$, and also by the condition that the function G is unceasing in its variable we get

$$G\big(\xi v, S(\xi v), S(\xi v)\big) = q \ max\big\{2G\big(\xi v, S(\xi v), S(\xi v)\big), G\big(\xi v, S(\xi v), S(\xi v)\big)\big\}$$

[4.3.9]

Since $0 \leq q \leq {}^1/_2$, which gives contradiction and therefore we have

$\xi v = \ S(\xi v).$

Uniqueness: Let us suppose that we have another point w as a invariant point then $\xi w \neq \xi v$ such that $S(\xi w) = \xi w$, then

$$G(\xi v, \xi w, \xi w) \leq q \ max \begin{Bmatrix} G(\xi v, \xi v, \xi w) + G(\xi w, \xi w, \xi v), \\ G(\xi w, \xi w, \xi w) + G(\xi w, \xi w, \xi w), \\ G(\xi w, \xi w, \xi v) + G(\xi v, \xi v, \xi w) \end{Bmatrix}$$

[4.3.10]

So, we deduce that

$$G(\xi v, \xi w, \xi w) \leq q[G(\xi v, \xi w, \xi w) + G(\xi w, \xi v, \xi v)]$$

This implies that

$$G(\xi v, \xi w, \xi w) \leq \frac{q}{1-q} \ G(\xi w, \xi v, \xi v)$$

And by repeated process of the same argument we will invent that

$$G(\xi v, \xi w, \xi w) \leq \frac{q}{1-q} \ G(\xi w, \xi v, \xi v)$$

Therefore we get

$$G(\xi v, \xi w, \xi w) \leq \left(\frac{q}{1-q}\right)^2 G(\xi w, \xi v, \xi v)$$

Since $0 < \frac{q}{1-q} < 1$,

Which gives contradiction then we say that $\xi v = \xi w$.

Also to prove that S is G- unceasing at ξv.

Suppose that $\{\xi y_n\}$ be sequence such that $\{\xi y_n\} \subset$ U and

$\lim_{n\to\infty} \xi y_n = \xi v$ in (U,G), then

$$G(S(\xi y_n), S(\xi v), S(\xi v)) \leq q\ max \begin{Bmatrix} G(\xi y_n, S(\xi y_n), S(\xi v)) + G(\xi v, S(\xi v), S(\xi y_n)), \\ G(\xi v, S\xi, S(\xi v)) + G(\xi v, S(\xi v), S(\xi v)), \\ G(\xi v, S(\xi v), S(\xi y_n)) + G(\xi y_n, S(\xi y_n), S(\xi v)) \end{Bmatrix}$$

[4.3.11]

And we have

$$G(S(\xi y_n), \xi\, v, S(\xi v)) = q\ max \begin{Bmatrix} G(\xi y_n, S(\xi y_n), S(\xi v)) + G(\xi v, S(\xi v), S(\xi y_n)), \\ 2G(\xi v, S(\xi v), S\xi(v)) \end{Bmatrix}$$

[4.3.12]

Thus

$$G(S(\xi y_n), \xi v, \xi v) \leq q[G(\xi y_n, S(\xi y_n), S(\xi v)) + G(\xi v, \xi v, S(\xi y_n))]$$

[4.3.13]

But by (G 5) we have

$$G(\xi y_n, S(\xi y_n), \xi v) \leq q[G(\xi y_n, \xi v, \xi v) + G(\xi v, \xi v, S(\xi y_n))]$$

Which implies that

$$G(S(\xi y_n), \xi\, v, \xi v) \leq q\ G(\xi y_n, \xi v, \xi v) + 2q\ G(\xi v, \xi v, S(\xi y_n))$$

That is we have

$$G(S(\xi y_n), \xi v, \xi v) \leq \frac{q}{1-2q} G(\xi y_n, \xi v, \xi v)$$

[4..14]

Therefore at limit $n \to \infty$ we have

$G(S(\xi y_n), \xi v, \xi v) \to 0$, so, by proposition we have

$S(\xi y_n) \to \xi v = S(\xi v)$

Which verifies that S is G- unceasing at ξv.

Corollary 4.4: Suppose a metric (U, G) space be random G- Complete or it is Complete random G-Metric then for a mapping S : U → U which satisfies the subsequent axioms for all $\xi x,\ \xi y, \xi z \in U$, then

$$G\big(S^p(\xi x), S^p(\xi y), S^p(\xi z)\big) \leq q\ max \begin{Bmatrix} G\big(\xi x, S^p(\xi x), S^p(\xi x)\big) + G\big(\xi y, S^p(\xi y), S^p(\xi x)\big), \\ G\big(\xi y, S^p(\xi y), S^p(\xi z)\big) + G\big(\xi z, S^p(\xi z), S^p(\xi y)\big), \\ G\big(\xi z, S^p(\xi z), S^p(\xi x)\big), G\big(\xi x, S^p(\xi x), S^p(\xi z)\big) \end{Bmatrix}$$

[4.4.1]

Where $q \in \left(0, {}^{1}\!/_{2}\right]$ then ξv is an unique fixed point of mapping S and S^p is G- continous at ξv.

Proof : Can be proved as privious theorem.

Theorem 4.7: Suppose a metric (U, G) space be Complete random G-metric then for a mapping S : U → U which gratifies the subsequent axioms for all $\xi x, \xi y, \xi z \in U$, then

$$G\big(S(\xi x), S(\xi y), S(\xi z)\big) \leq q\ max \begin{Bmatrix} G\big(\xi z, S(\xi z), S(\xi z)\big) + G\big(\xi x, S(\xi z), S(\xi z)\big), \\ G\big(\xi z, S(\xi y), S(\xi y)\big) + G\big(\xi x, S(\xi y), S(\xi y)\big), \\ \big[2G\big(\xi y, S(\xi x), S(\xi x)\big)\big], \big[2G\big(\xi z, S(\xi x), S(\xi x)\big)\big] \end{Bmatrix}$$

[4.7.1]

Where $q \in \left(0, {}^{1}\!/_{3}\right]$ then v is a exclusive invariant point of mapping S and S is random G- continuous at ξ v.

Proof: Assume that S setisfies the subsequent axioms and let ξx_0 be any random point in U, then we express a classification $\{\xi x_n\}$ in U such that

$$\xi x_n = S^n(\xi x_0)$$

Then from [8.1.7.1], we have

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq q\ max \begin{Bmatrix} G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) + G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1}), \\ G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) + G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1}), \\ [2G(\xi x_n, \xi x_n, \xi x_n)], [2G(\xi x_n, \xi x_n, \xi x_n)] \end{Bmatrix}$$

[4.7.2]

$$= q\ max\{G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) + G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1})\}$$

Thus

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) < q\ G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) + q\ G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1})$$

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) < \frac{q}{1-q} G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1})$$

[4.7.3]

Therefore by property G (5) we have

$$G(\xi x_{n-1}, \xi x_{n+1}, \xi x_{n+1}) \leq G(\xi x_{n-1}, \xi x_n, \xi x_n) + G(\xi x_n, \xi x_{n+1}, \xi x_{n+1})$$

[4.7.4]

Let $k = \frac{q}{1-2q}$, then $q \in [0, {}^{1}/_{3})$ and from eq. [8.1.7.3]

We forms a result that

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq k\ G(\xi x_{n-1}, \xi x_n, \xi x_n)$$

[4.7.5]

Now by repeated process we get

$$G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) \leq k^n\ G(\xi x_0, \xi x_1, \xi x_1)$$

Then for all $n, m \in N, n < m$ we have by continuing use of rectangular inequality that

$$G(\xi x_n, \xi x_m, \xi x_m) \leq G(\xi x_n, \xi x_{n+1}, \xi x_{n+1}) +$$
$$G(\xi x_{n+1}, \xi x_{n+2}, \xi x_{n+2}) +$$
$$G(\xi x_{n+2}, \xi x_{n+3}, \xi x_{n+3}) +$$
$$\dots \dots \dots \dots$$
……….+ $G(\xi x_{m-1}, \xi x_m, \xi x_m)$

$$\leq (k^n + k^{n+1} + \cdots \dots \dots \dots \dots + k^{m-1})\ \ G(\xi x_0, \xi x_1, \xi x_1)$$

$$\leq \frac{k^n}{1-k}\ \ G(\xi x_0, \xi x_1, \xi x_1)$$

Then, $\lim_{n,m\to\infty} G(\xi x_n, \xi x_m, \xi x_m) = 0, n, m \to \infty$

We say the classification $\{\xi x_n\}$ is a random G- Cauchy classification. By the condition that (U,G) is complete, then we have for $\xi\, v \in U$ there exist a classification $\{\xi x_n\}$ is G- convergent to ξv.

Let us assume that S($\xi v) \neq \xi v$, then

$$G(\xi x_n, S(\xi v), S(\xi v)) \leq q\ max \begin{Bmatrix} G\big(\xi v, S(\xi v), S(\xi v)\big) + G\big(\xi x_{n-1}, S(\xi v), S(\xi v)\big), \\ G\big(\xi v, S(\xi v), S(\xi v)\big) + G\big(\xi x_{n-1}, S(\xi v), S(\xi v)\big), \\ [2G(\xi v, \xi x_n, \xi x_n)], [2G(\xi v, \xi x_n, \xi x_n)] \end{Bmatrix}$$

[4.7.6]

Now taking limit as $n \to \infty$ and by the condition that the function G is continuous in its variables, we have

$$G\big(\xi v, S(\xi v), S(\xi v)\big) \leq 2q\ G\big(\xi v, S(\xi v), S(\xi v)\big)$$

Since $0 \leq q \leq {}^{1}/_{3}$, which gives contradiction and therefore we have

$\xi v = S(\xi v)$.

Uniqueness: Let us assume that we have another point ξw as a invariant point then $\xi w \neq \xi v$ such that $S(\xi w) = \xi w$, then

$$G(\xi v, \xi w, \xi w) \leq q\ max \begin{Bmatrix} G(\xi w, \xi w, \xi w) + G(\xi v, \xi w, \xi w), \\ G(\xi w, \xi w, \xi w) + G(\xi v, \xi w, \xi w), \\ [2G(\xi w, \xi v, \xi v)], [G(\xi w, \xi v, \xi v)] \end{Bmatrix}$$

[4.7.7]

Thus

$$G(\xi v, \xi w, \xi w) \leq qmax\ [G(\xi v, \xi w, \xi w), 2G(\xi w, \xi v, \xi v)]$$

And we prove that

$$G(\xi v, \xi w, \xi w) \leq 2q\ G(\xi w, \xi v, \xi v)$$

[4.7.8]

Now by the same argument, we get

$$G(\xi w, \xi v, \xi v) \leq 2q\ G(\xi v, \xi w, \xi w)$$

[4.7.9]

Hence, $G(\xi v, \xi w, \xi w) \leq 4q^2\ G(\xi v, \xi w, \xi w)$

Which gives that $\xi v = \xi w$.

Since $0 \leq q \leq {}^{1}/_{3}$, then for $0 \leq 4q^2 < 1$.

Now to verify that S is random G- continuous at ξv.

Assuming us we suppose a sequence $\{\xi y_n\} \subseteq U$ such that

$\lim \xi y_n = \xi v$, then

$$G(S(\xi v), S(\xi y_n), S(\xi y_n)) \leq q\ max \begin{Bmatrix} G\big(\xi y_n, S(\xi y_n), S(\xi y_n)\big) + G\big(\xi v, S(\xi y_n), S(\xi y_n)\big), \\ G\big(\xi y_n, S(\xi y_n), S(\xi y_n)\big) + G\big(\xi v, S(\xi y_n), S(\xi y_n)\big), \\ \big[2G\big(\xi y_n, S(\xi v), S(\xi v)\big)\big], \big[2G\big(\xi y_n, S(\xi v), S(\xi v)\big)\big] \end{Bmatrix}$$

[4.7.10]

Therefore by eq. [4.7.10]

We have two cases

Case-I $G(\xi v, S(\xi y_n), S(\xi y_n)) \leq 2q\ G(\xi y_n, \xi v, \xi v)$

Case-II

$G(\xi v, S(\xi y_n), S(\xi y_n)) \leq$
$\left(\frac{q}{1-q}\right) G\left(\xi y_n, S(\xi y_n), S(\xi y_n)\right)$

But , by (G 5) we have

$$G\left(y_n, S(y_n), S(y_n)\right) \leq G(y_n, v, v) + G(v, S(y_n), S(y_n))$$

So Case II we have

$$G(\xi v, S(\xi y_n), S(\xi y_n)) \leq k\, G(\xi y_n, \xi v, \xi v)$$

In every case taking the limit $\rightarrow \infty$, we have

$G(\xi v, S(\xi y_n), S(\xi y_n)) \rightarrow 0$ and so,

So by proposition we have

$$S(\xi y_n) \rightarrow \xi v = S\xi v$$

This proves that S is random G- continous at ξ v.

Corollary 4.8: Suppose a metric (U, G) space be Complete random G-Metric then for a mapping S : U → U which gratifies the subsequent axioms for all $\xi x, \xi y, \xi z \in U$, then

$$G\left(S^p(\xi x), S^p(\xi y), S^p(\xi z)\right) \leq q\, max \left\{ \begin{array}{c} G\left(\xi z, S^p(\xi z), S^p(\xi z)\right) + G\left(\xi x, S^p(\xi z), S^p(\xi z)\right), \\ G\left(\xi z, S^p(\xi y), S^p(\xi y)\right) + G\left(\xi x, S^p(\xi y), S^p(\xi y)\right), \\ \left[2G\left(\xi y, S^p(\xi x), S^p(\xi x)\right)\right], \left[2G\left(\xi z, S^p(\xi x), S^p(\xi x)\right)\right] \end{array} \right\}$$

[4.8.1]

Where $q \in [0, {}^1/_3)$ then v is a exclusive invariant point of mapping S and S^p is G- continuous at v.

Proof: This corollary can be proved easily as previous theorem.

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